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Construction of the irreducibles of $B(2, 2)$

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Abstract
The ordinary Bondi–Metzner–Sachs (BMS) group B is the common asymptotic symmetry group of all radiating, asymptotically flat, Lorentzian spacetimes. As such, B is the best candidate for the universal symmetry group of general relativity. However, in studying quantum gravity, spacetimes with signatures other than the usual Lorentzian one and complex spacetimes are frequently considered. Generalizations of B appropriate to these other signatures have been defined earlier. In particular, the generalization $B(2, 2)$ appropriate to the ultrahyperbolic signature $(+, +, -, -)$ has been described in detail, and the study of its irreducible unitary representations (IRs) of $B(2, 2)$ has been initiated. The infinite little groups have been given explicitly, but the finite little groups have only been partially described. This study is completed by describing in detail the finite little groups and by giving all the necessary information in order to construct the IRs of $B(2, 2)$ in all cases.

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1. Introduction

In a study of the class of asymptotically flat spacetimes which represent bounded gravitational sources that are quiet in the beginning, then emit gravitational radiation, and then turn quiet again, Bondi, Metzner and Sachs found [1] that the set of coordinate transformations which preserve suitable, radiation-dictated, boundary conditions imposed on the asymptotic region in light-like future directions, forms a pseudo-group of local diffeomorphisms ('asymptotic isometries'), the BMS group B However, Penrose showed that [2], by 'going to infinity', B could be interpreted as an *exact* global transformation group $B \times I^+ \longrightarrow I^+$ of the 'future null boundary' I^+ of the spacetimes concerned. Due to its universality—it is the same for the whole class of asymptotically flat spacetimes studied by Bondi, Metzner and Sachs—B soon attracted attention as an approach to quantum gravity [3, 4] or the problem of 'internal symmetries' [5]. A study of IRs of B was started by Sachs [3], and taken further by Cantoni [6].

In a study [7] whose importance is difficult to overemphasize and which ever since has shaped to a great extent the spirit of research in elementary particle physics, Wigner (in 1939) treated explicitly for the first time the infinite-dimensional representations of a Lie group, namely of the Poincare group P. In this study, Wigner, using a minimal number of well-established physical principles, gave among other things (a) a complete *unconstrained* description of *all* possible solutions of *all* possible Poincare-invariant wave equations without finding or solving the equations, and (b) a theoretical definition of a relativistic elementary particle—it is that physical entity which is described by a unitary irreducible representation of the Poincare group P. Moreover, Wigner isolated within representation theory a parameter as being the mathematical counterpart of the 'spin' of the elementary particle; this parameter is one of the two parameters which parameterize the IRs of P, the second one being the mass of the elementary particle squared. Wigner's ideas have been repeatedly exploited and generalized in mathematics, particularly, by G W Mackey, in developing the theory of induced representations of locally compact groups [8–10].

McCarthy found it reasonable to attempt to lay a firm foundation for quantum gravity by following through Wigner's programme with B replacing P. For this purpose, he constructed in a series of papers [11–13] the IRs of B by using Wigner–Mackey's representation theory. It turns out that there are two striking differences between P and B. The first difference is as follows. The 'little groups' for P (from which the IRs are induced) are *not* all compact. Compact little groups always give discrete spins, whereas, the non-compact ones also give *continuous* spins. However, the little groups for B are always compact [12], and this means that all the IRs of B necessarily have only *discrete* spins. It is as though the presence of gravity obstructs the unphysical continuous spins of special relativity; that is, gravity gives a possible explanation for the observed discreteness of elementary particle spins.

The second difference is the following. While the little groups of P are all of *infinite* order, some of the little groups of B are *finite*. Indeed, certain IRs of B are induced [12] from the IRs of the finite symmetry groups of the planar regular polygons or of the platonic solids in ordinary Euclidean 3-space; that is, some of the IRs of B are induced from the complex linear IRs of the cyclic or dihedral groups, or symmetry groups of the tetrahedron, cube or icosahedron [12]. The much later appearance of precisely the same *complex linear* IRs of precisely the same finite groups (and not just the groups themselves!) in gravitational instanton theory (which is concerned with the *real nonlinear* self-dual Euclidean Einstein equations) suggests [17] a connection with the IRs of B. For details of gravitational instanton theory, see, e.g., [14, 15].

The role of the IRs of the group B is, however, much less well understood [16] than the role of Wigner's IRs. In order to make this role better understood and in order to make connections with other approaches to quantum gravity, where Euclidean or complex versions of general relativity are frequently considered, McCarthy showed [17] that B admits 42 generalizations to real spacetimes of any signature, and also to complex spacetimes. It is an amazing and totally unexpected result that all the 'little groups' for the complex BMS group CB [17], the Euclidean BMS group EB [17], and the ultrahyperbolic BMS group $B(2, 2)$ [19], the generalizations of B appropriate to complex, Euclidean and ultrahyperbolic spacetime respectively, are compact, which means that all the CB-elementary entities, all the EB-elementary entities, as well as all the $B(2, 2)$-elementary entities carry discrete spin. Kronheimer [14, 15] has given a complete classification of instanton moduli spaces for *Euclidean* instantons. However, his description only partially describes the moduli spaces, since it still involves *constraints*. Kronheimer does not solve the constraint equations, but it has been argued [17, 18] that the IRs of the BMS group and of its generalizations in complex spacetimes as well as in spacetimes with Euclidean or ultrahyperbolic signature are what really lie behind the full description of (unconstrained) moduli spaces of gravitational instantons.

The generalization $B(2, 2)$ of B appropriate to the ultrahyperbolic signature $(+, +, -, -)$ has been described in detail [17], and the study of the IRs of $B(2, 2)$ has been initiated [19] (see [20] for a non-group-theoretic approach to ultrahyperbolic general relativity). Interestingly enough *finite* little groups appear not only in the study of B but also in the study of $B(2, 2)$ [19]. In [19] the *infinite* little groups have been given explicitly but the *finite* little groups have only been partially described: they are those subgroups of the Cartesian product group $C_n \times C_m$ which contain the element $(-I, -I)$, where C_r is the cyclic group of order r, r being finite, and I is the identity element. Therefore, the problem of constructing the *finite* little groups reduces [19] to a seemingly very simple task; that of classifying all subgroups of $C_n \times C_m$. Surprisingly, this task is less simple than it may appear at first sight. It turns out that the solution is constructed from the 'fundamental cases' $n = p^a$, $m = p^\beta$, (n, m are powers of the same prime), via the prime decomposition of m and n. In this paper, we complete the study of IRs of $B(2, 2)$ initiated in [19] by giving explicit expressions of the finite little groups of $B(2, 2)$ and by providing all the necessary information in order to construct the operators of the induced representations of $B(2, 2)$ in all cases. Quite apart from the possible link with instantons, the study of the representation theory of $B(2, 2)$, as well as of the other generalizations of B, is of much independent interest. Indeed, the results of this study are part of a direct and reliable group-theoretic approach to quantum gravity.

In section 2 a summary of the results on the representation theory of $B(2, 2)$ obtained so far is given. In section 3 it is shown that the problem of constructing the subgroups of $C_n \times C_m$ is reduced to the problem of constructing the subgroups of $C_{p^a} \times C_{p^\beta}$, where p is a prime number and a, β are non-negative integers. In section 4 the cyclic subgroups of $C_{p^a} \times C_{p^\beta}$ are determined. In section 5 the non-cyclic subgroups of $C_{p^a} \times C_{p^\beta}$ are counted. In section 6 the subgroups of $C_{p^a} \times C_{p^\beta}$ with two generators are determined. In section 7 explicit expressions for the generators of the subgroups of $C_n \times C_m$ are given. In section 8 generators of the finite little groups of $B(2, 2)$ are determined explicitly. In section 9 all the necessary information in order to construct explicitly the operators of the induced representations of $B(2, 2)$ is given.

2. Results on the representation theory of $B(2, 2)$

The induced representations of $B(2, 2)$ are constructed by using Wigner–Mackey's representation theory of semi-direct products. A summary of Wigner–Mackey's theory is presented in appendix A. We summarize now the results obtained so far [19] on the representation theory of $B(2, 2)$. The group $B(2, 2)$ is the BMS group appropriate to the 'ultrahyperbolic' signature and asymptotic flatness in null directions [17]. Recall that the ultrahyperbolic version of Minkowski space, sometimes written as $R^{2,2}$, is just R^4—the vector space of row vectors with 4 real components—with scalar product $x \cdot y$ between x and y given by

$$x \cdot y = x^0 y^0 + x^2 y^2 - x^1 y^1 - x^3 y^3, \tag{1}$$

where $x, y \in R^4$ have components x^μ and y^μ respectively, where $\mu = 0, 1, 2, 3$. The group $B(2, 2)$ is given by

$$B(2, 2) = L_e^2(T^2) \circledS_T G^2 \tag{2}$$

where the homomorphism T from G^2 into the group of automorphisms of $L_e^2(T^2)$ is given by

$$(T(g,h)\alpha)(m,n) = k(m,g)k(n,h)\alpha([mg],[nh]), \tag{3}$$

where $(g,h) \in G^2 = G \times G$ ($G = SL(2,R)$ is the matrix Lie group of 2×2 matrices with real entries and unit determinant) and $\alpha \in L_e^2(T^2)$; $L_e^2(T^2)$ is the Hilbert space of all even square integrable functions defined on T^2, and $T^2 = S^1 \times S^1$ is the 2-torus. We specify now the meaning of evenness involved here. Let $Sp = R^2 - 0$ be the Cartesian plane punctured at the origin. In analogy with the Lorentzian case, the null cone in $R^{2,2}$ is the set of vectors with zero length. We define $\mathcal{N} \subset R^{2,2}$ to be the null cone with the origin deducted: $\mathcal{N} = \{x \in R^{2,2} \mid x \neq 0, x\cdot x = 0\}$. With each vector $x = (x_1, x_2) \in Sp$, we associate its length $r \equiv |x| \equiv \sqrt{x_1^2 + x_2^2}$ and the unit length vector $m \equiv [x] \equiv x/|x|$ having the same direction as x. Thus we have

$$x = rm, \qquad r = |x|, \qquad m \equiv [x] \equiv x/|x|. \tag{4}$$

Let $S^1 \subset Sp$ be the set of vectors of unit length in $Sp : S^1 = \{x \in Sp \mid |x| = 1\}$. Each factor of $T^2 = S^1 \times S^1$ is given by the last equality. If $(x,y) \in Sp^2 \equiv Sp \times Sp$, define the radius and direction of x by equation (4), and the radius t and direction n of $y = (y_1, y_2)$ by

$$y = tn, \qquad t = |y|, \qquad n \equiv [y] \equiv y/|y|. \tag{5}$$

The set of all real-valued functions $\alpha : T^2 \to R$, $\alpha \in L_e^2(T^2)$ are *even*; that is, they satisfy the evenness condition $\alpha(-m,-n) = \alpha(m,n)$. It turns out that the k-factors which appear in (3) are given by $k(m,g) = |mg|$, and similarly, $k(n,h) = |nh|$. Finally, $[mg] \equiv (xg)/|xg|$ and $[nh] \equiv (yh)/|yh|$, where xg, yh denote multiplication from the right of the row vectors x, h with the $SL(2,R)$ matrices g and h respectively. Using the last expressions for the k-factors one can readily prove that the map T defined by (3) is an homomorphism. The usual angular coordinates (ρ,σ) for T^2 are defined by

$$m = (\cos\rho, \sin\rho) \qquad \text{so} \qquad x_1 = r\cos\rho, \quad x_2 = r\sin\rho, \tag{6}$$

$$n = (\cos\sigma, \sin\sigma) \qquad \text{so} \qquad y_1 = t\cos\sigma, \quad y_2 = t\sin\sigma, \tag{7}$$

where ρ and σ are the real numbers defined mod 2π, which may be taken in the ranges $0 \leqslant \rho < 2\pi$, $0 \leqslant \sigma < 2\pi$. The Lesbegue measure is defined by the 2-form

$$\mathrm{d}\lambda(\tau) = \mathrm{d}\rho \wedge \mathrm{d}\sigma \tag{8}$$

where $\tau = (m,n)$. $L_e^2(T^2)$ becomes a real Hilbert space via the introduction of the scalar product

$$\langle\alpha,\beta\rangle = \int_{T^2} \alpha(\tau)\beta(\tau)\,\mathrm{d}\lambda(\tau). \tag{9}$$

The norm defined by the scalar product (9) induces a metric in $L_e^2(T^2)$. The balls defined by this metric define the open sets of a topology in $L_e^2(T^2)$ which is called Hilbert topology. With this topology, $L_e^2(T^2)$, in addition to being a real Hilbert space, becomes an Abelian topological group. The irreducible unitary continuous representations of $L_e^2(T^2)$ (characters) can be given the structure of an Abelian group $\widehat{L}$, the dual group of $L_e^2(T^2)$. The action T of G^2 on $L_e^2(T^2)$ induces a dual action $\widehat{T}$ of G^2 on $\widehat{L}$ defined by

$$(\widehat{T}(g,h)\chi)(\alpha) := \chi(T((g,h)^{-1})\alpha), \tag{10}$$

where, $\alpha \in L_e^2(T^2)$, $\chi \in \widehat{L}$, and $(g, h) \in G^2$. As it is explained in appendix A it is this dual action $\widehat{T}$ which, in principle, determines the IRs of $B(2, 2)$. However, the mathematical treatment is facilitated by passing to another action on the space of functionals of $L_e^2(T^2)$. In fact, it can be shown [11] that *every* character χ can be written *uniquely* as

$$\chi(\alpha) = \mathrm{e}^{\mathrm{i}(\phi,\alpha)}, \tag{11}$$

where $\alpha \in L_e^2(T^2)$, ϕ is a continuous linear functional on $L_e^2(T^2)$; that is, ϕ belongs to the topological dual $L_e^{2'}(T^2)$ of $L_e^2(T^2)$, and (ϕ, α) denotes the value of the functional ϕ on the element α. From the Reisz–Fréchet theorem for Hilbert spaces one concludes that given a continuous linear functional $\phi \in L_e^{2'}(T^2)$, we can write for $\alpha \in L_e^2(T^2)$

$$(\phi, \alpha) = \langle \phi, \alpha \rangle, \tag{12}$$

where the *function* $\phi \in L_e^2(T^2)$ on the right-hand side is uniquely determined by (and denoted by the same symbol as) the linear functional $\phi \in L_e^{2'}(T^2)$ on the left-hand side. Equation (11) yields that $\widehat{L}$ is isomorphic (as an Abelian group) to the topological dual $L_e^{2'}(T^2)$, which in turn, via equation (12), is isomorphic (as a Hilbert space) to the space of even functions $L_e^2(T^2)$. The dual action $\widehat{T}$ of G^2 on $\widehat{L}$ induces an action T' of G^2 on $L_e^{2'}(T^2)$ defined by

$$(\widehat{T}(g, h)\chi)(\alpha) := \mathrm{e}^{\mathrm{i}\langle T'(g,h)\phi,\alpha\rangle}. \tag{13}$$

Compatibility of equations (10) and (13) requires

$$\langle T'(g, h)\phi, \alpha\rangle = \langle \phi, T((g, h)^{-1})\alpha\rangle. \tag{14}$$

It is precisely this induced action T' on the space of functionals of $L_e^2(T^2)$ which determines, in practice, the structure of IRs of $B(2, 2)$. A straightforward calculation shows that

$$(T'(g, h)\phi)(m, n) = k^{-3}(m, g)k^{-3}(n, h)\phi([mg], [nh]), \tag{15}$$

where $\phi(m, n) \in L_e^2(T^2)$. Recall (see appendix A) that the 'little group' L_ϕ of any $\phi \in L_e^2(T^2)$ is the *largest* subgroup of G^2 which satisfies

$$L_\phi = \{(g, h) \in G^2 \mid T'(g, h)\phi = \phi.\}. \tag{16}$$

Equation (13) implies ((cf equation (A.4)) that

$$L_\phi = L_\chi, \qquad \text{where} \qquad L_\chi = \{(g, h) \in G^2 \mid \widehat{T}(g, h)\chi = \chi\}. \tag{17}$$

Henceforth, for convenience, since $L_\chi = L_\phi$, we will employ the group L_ϕ defined in (16). $L_e^2(T^2)$ has been endowed with the Hilbert topology and $G = SL(2, R)$ is equipped with the standard topology given by the metric derived from the norm

$$|g| = (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{1/2}, \qquad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G. \tag{18}$$

In the product topology of $L_e^2(T^2) \times G^2$, $B(2, 2)$ becomes a topological group and it is in this topology that all the 'little groups' are compact (in a finer topology non-compact little groups may arise [24]). For every little group L_ϕ we define the corresponding *invariant subspace* $L_e^2(L_\phi) \subset L_e^2(T^2)$ by

$$L_e^2(L_\phi) = \left\{\phi \in L_e^2(T^2) \,\middle|\, T'(l)\phi = \phi \text{ for all } l \in L_\phi\right\}. \tag{19}$$

Henceforth, $R(\vartheta)$ denotes $\left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)$. Moreover, $\widetilde{\phi}(\rho, \sigma)$ is defined by

$$\phi = \phi(m, n) = \phi(m(\rho), n(\sigma)) = \widetilde{\phi}(\rho, \sigma). \tag{20}$$

The results are summarized in the following table:

	Little groups L_ϕ and the corresponding invariant spaces $L^2_e(L_\phi)$	
	L_ϕ	$L^2_e(L_\phi)$
(1)	$SO(2)\times SO(2)=(R(\vartheta),R(\varphi))$	$\widetilde{\phi}(\rho,\sigma)=c,$ some $c\in R$
(2)	$C_N\times SO(2)=\left(R\left(\frac{2\pi}{N}i\right),R(\varphi)\right),$ where N is even.	$\widetilde{\phi}(\rho,\sigma)=g(\rho)$ $g(\rho)$ is periodic of period $\frac{2\pi}{N}$ $\int_0^{\frac{2\pi}{N}}(g(\rho))^2\,\mathrm{d}\rho<+\infty$
(3)	$SO(2)\times C_N=\left(R(\vartheta),R\left(\frac{2\pi}{N}i\right)\right),$ where N is even.	$\widetilde{\phi}(\rho,\sigma)=l(\sigma)$ $l(\sigma)$ is periodic of period $\frac{2\pi}{N}$ $\int_0^{\frac{2\pi}{N}}(l(\sigma))^2\,\mathrm{d}\rho<+\infty$
(4)	$H(N,p,q)=\left(R(p\vartheta),R\left(q\vartheta+\frac{2\pi}{N}i\right)\right)$, where, either, both p and q are odd, $p/N=p'/N'$, where p',N' are coprime, or, p and q have opposite parity, $p/N=p'/N'$, where p',N' are coprime, and, N' is even.	$\widetilde{\phi}(\rho,\sigma)=f(p\sigma-q\rho)\equiv f(\widehat{\sigma})$ $f(\widehat{\sigma})$ is periodic of period $\frac{2\pi}{N'}$ $\int_0^{\frac{2\pi}{N'}}(f(\widehat{\sigma}))^2\,\mathrm{d}\widehat{\sigma}<+\infty$
(5)	Subgroups $\mathcal{C}$ of $C_n\times C_m$ which contain the element $(R(\pi),R(\pi))=(-I,-I)$. Both n and m are finite and even.	$\widetilde{\phi}(\rho,\sigma)=0\mid(\rho,\sigma)\notin E_{\mathcal{C}}$ $\int_{E_{\mathcal{C}}}(\widetilde{\phi}(\rho,\sigma))^2\,\mathrm{d}\rho\wedge\mathrm{d}\sigma<+\infty$

The description of the invariant subspaces $L^2_e(L_\phi)$ of the fifth class of groups needs some explanation. The representation (15), when restricted to $SO(2)\times SO(2)$ and expressed in terms of ρ,σ, becomes

$$(T'(R(\vartheta),R(\varphi))\widetilde{\phi})(\rho,\sigma)=\widetilde{\phi}(\rho+\vartheta,\sigma+\varphi) \tag{21}$$

for all ρ,σ and all ϑ,φ. Therefore, the representation T' dictates a fixed point free action $T^2\times(SO(2)\times SO(2))\to T^2$ given by

$$((\rho,\sigma),(R(\vartheta),R(\varphi))\longmapsto(\rho+\vartheta,\sigma+\varphi). \tag{22}$$

When the last action is restricted to a (finite) subgroup $\mathcal{C}$ of $C_n\times C_m$ it reads

$$((\rho,\sigma),(g_i,g_j))\longmapsto(\rho g_i,\sigma g_j) \tag{23}$$

where $0\leqslant i\leqslant(n-1), 0\leqslant j\leqslant(m-1)$, $g_i\in C_n, g_j\in C_m$ and $\rho g_i=\rho+\frac{2\pi}{n}i, \phi g_j=\sigma+\frac{2\pi}{m}j$. The set $E_{\mathcal{C}}\subset T^2$ is an elementary domain for the subgroup $\mathcal{C}$ for precisely this action. Recall that an *elementary domain* E for an action of a finite group G on a topological space M is an open subset $E\subset M$ such that the following conditions are satisfied:

$$\begin{aligned}&\text{(A)}\quad \text{For any } g_1,g_2\in G, \text{ with } g_1\neq g_2,\ Eg_1\cap\overline{E}g_2=\emptyset,\\ &\text{(B)}\quad \bigcup_{g\in G}\overline{E}g=M \text{ (disjoint union).}\end{aligned} \tag{24}$$

Here bar means topological closure, and $\emptyset$ means the empty set. It is easy to show that the open set $F_{nm}\subset T^2$ defined by the formula $F_{nm}=E_n\times E_m\subset T^2$, where E_n is the open set $E_n=\{\theta\in S^1\mid 0<\theta<2\pi/n\}$, is an elementary domain for the action (23). Interestingly enough, it can be shown [19] that there is a relation between $E_{\mathcal{C}}$ and F_{nm} : $E_{\mathcal{C}} = F_{nm}S$, where S is *any* selection of representatives of left cosets of the coset space $(C_n \times C_m)/\mathcal{C}$. From every pair $(U, \widetilde{\phi})$, where U is a unitary irreducible representation of L_ϕ and $\widetilde{\phi}$ is a specific element of the corresponding invariant space $L_e^2(L_\phi)$, a unique (up to equivalence) unitary irreducible representation of $B(2, 2)$ can be constructed (see appendix A). In the case of the *finite* little groups the determination of $\widetilde{\phi}$ requires the description of the domains $E_{\mathcal{C}}$ of the functions $\widetilde{\phi}$, which in turn necessitates the *explicit* determination of the finite little groups of $B(2, 2)$, i.e., of the subgroups $\mathcal{C}$ of $C_n \times C_m$ which contain the element $(-I, -I)$ [19]. The next six sections are devoted to the task of determining them explicitly. In sections 3–7 the subgroups of $C_n \times C_m$ are constructed. In section 8 the specific subgroups $\mathcal{C}$ which contain the element $(-I, -I)$ are isolated.

3. The problem simplified

In this section, we show that the problem of constructing the subgroups of $C_n \times C_m$ can be simplified: it is reduced to the problem of constructing the subgroups of $C_{p^a} \times C_{p^\beta}$, where p is a prime number and a, β are non-negative integers. This is precisely the content of the proposition which follows.

Proposition 1. *Let $C_n \times C_m$ be the direct product of the cyclic groups of finite order C_n and C_m. Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ and $m = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}$ be the prime decomposition of the integers n and m, i.e., $p_i, i = 1, 2, \ldots, s$, are distinct prime numbers and a_i, β_i are non-negative integers. Any subgroup of $C_n \times C_m$ has the form*

$$C_{q_1^{\lambda_1}} \times C_{q_2^{\lambda_2}} \times \cdots \times C_{q_\sigma^{\lambda_\sigma}}, \tag{25}$$

i.e., is a direct product where the numbers $q_1, q_2, \ldots, q_\sigma$ are prime and each one of them appears at most twice. For any q_j, $j = 1, 2, \ldots, \sigma$, there exists a p_i, $i = 1, 2, \ldots, s$, so that $q_j = p_i$. When q_j appears once $\lambda_j \in [1, \max(a_i, \beta_i)]$. When q_j occurs twice, say $q_j = q_{j+k}$, then one of the indices λ_j, λ_{j+k} belongs to $[1, a_i]$ and the other one belongs to $[1, \beta_i]$. For every subgroup of $C_n \times C_m$ the expression (25) is unique.

Proof. Let $n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_s^{r_s}$ and $m = p_1^{t_1} \cdot p_2^{t_2} \cdots p_s^{t_s}$ be the prime decomposition of the integers n and m; i.e., p_i, $i = 1, 2, \ldots, s$, are distinct prime numbers and r_i, t_i are non-negative integers. We have

$$C_n \times C_m = \left(C_{p_1^{a_1}} \times C_{p_1^{\beta_1}}\right) \times \left(C_{p_2^{a_2}} \times C_{p_2^{\beta_2}}\right) \times \left(C_{p_3^{a_3}} \times C_{p_3^{\beta_3}}\right) \times \cdots \times \left(C_{p_s^{a_s}} \times C_{p_s^{\beta_s}}\right). \tag{26}$$

The group $C_n \times C_m$ is an Abelian group, and therefore, Sylow's second theorem (see, for example, [21], pages 128, 131 and 137) implies that the group $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i = 1, 2, \ldots, s$ is the unique Sylow p_i subgroup of $C_n \times C_m$ of order $p_i^{a_i+\beta_i}$. Every finite Abelian group is a direct product of primary cyclic groups [22]. Consequently, if A is a subgroup of $C_n \times C_m$ then $A = C_{q_1^{\lambda_1}} \times C_{q_2^{\lambda_2}} \times C_{q_3^{\lambda_3}} \times \cdots \times C_{q_\sigma^{\lambda_\sigma}}$, where $q_1, q_2, q_3, \ldots, q_\sigma$ are prime numbers, not necessarily distinct from one another, and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_\sigma$ are positive integers. According to Lagrange's theorem ${q_1}^{\lambda_1}{q_2}^{\lambda_2}{q_3}^{\lambda_3} \cdots {q_\sigma}^{\lambda_\sigma}$ divides ${p_1}^{a_1+\beta_1}\,{p_2}^{a_2+\beta_2}\,{p_3}^{a_3+\beta_3} \cdots {p_s}^{a_s+\beta_s}$. Therefore, $s \geqslant \sigma$ and for every q_j, $j = 1, 2, \ldots, \sigma$, there is a p_i, $i = 1, 2, \ldots, s$, so that $q_j = p_i$. Consider now all occurrences of p_i in A, so say $q_i = q_j = q_k = \cdots = q_w = p_i$. The group $A_s = C_{q_i^{\lambda_i}} \times C_{q_j^{\lambda_j}} \times C_{q_k^{\lambda_k}} \times \cdots \times C_{q_w^{\lambda_w}} \equiv C_{p_i^{\lambda_i}} \times C_{p_i^{\lambda_j}} \times C_{p_i^{\lambda_k}} \times \cdots \times C_{p_i^{\lambda_w}}$ is a p-subgroup of $C_n \times C_m$. Since $C_n \times C_m$ is an Abelian group Sylow's third theorem implies that $A_s = C_{p_i^{\lambda_i}} \times C_{p_i^{\lambda_j}} \times C_{p_i^{\lambda_k}} \times \cdots \times C_{p_i^{\lambda_w}}$ is a subgroup of the unique Sylow p_i subgroup of $C_n \times C_m$, $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$. The group $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$ is a finite Abelian group, and therefore the rank of any of its subgroups cannot be higher than its own rank. Therefore, since $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$ has two generators, there are at most two factors in $A_s = C_{p_i^{\lambda_i}} \times C_{p_i^{\lambda_j}} \times C_{p_i^{\lambda_k}} \times \cdots \times C_{p_i^{\lambda_w}}$. If A_s has one factor then $A_s = C_{p_i^{\lambda_j}}$, and $\lambda_j \in [1, \max(a_i, \beta_i)]$. If A_s has two factors then $A_s = C_{p_i^{\lambda_j}} \times C_{p_i^{\lambda_w}}$ and one of the two integers λ_j, λ_w, say the integer λ_j, belongs to the interval $[1, a_i]$, whereas the other integer λ_w belongs to the interval $[1, \beta_i]$. In particular ([23], page 42, theorem 3.3.3) if $\lambda_j \leqslant \lambda_w$ and if $a_i \leqslant \beta_i$ then $\lambda_j \in [1, a_i]$ and $\lambda_w \in [1, \beta_i]$. Suppose now that expression (25) is not unique. Thus, we assume that there are cyclic subgroups $C'_{q_1^{\lambda_1}}, C'_{q_2^{\lambda_2}}, C'_{q_3^{\lambda_3}}, \ldots, C'_{q_\sigma^{\lambda_\sigma}}$ of $C_n \times C_m$, such that

$$A = C'_{q_1^{\lambda_1}} \times C'_{q_2^{\lambda_2}} \times C'_{q_3^{\lambda_3}} \times \cdots \times C'_{q_\sigma^{\lambda_\sigma}}. \tag{27}$$

The prime q_j, $j \in \{1, 2, 3, \ldots, \sigma\}$, appears both in (25) and in (27) either once or twice. We examine firstly the case where q_j appears once. Then, the groups $C_{q_j^{\lambda_j}}$ and $C'_{q_j^{\lambda_j}}$ are Sylow q_j subgroups of group A. From Sylow's second theorem we conclude that the groups $C_{q_j^{\lambda_j}}$ and $C'_{q_j^{\lambda_j}}$ are conjugate. Since group A is Abelian, $C_{q_j^{\lambda_j}} = C'_{q_j^{\lambda_j}}$. The last equation holds for *every* prime number q_j, $j \in \{1, 2, 3, \ldots, \sigma\}$, which appears once in (25) (and in (27)). Secondly, we examine the case where the prime number q_j, $j \in \{1, 2, 3, \ldots, \sigma\}$, appears twice in (25) (and in (27)). Thus we have $q_j = q_{j+k}$, for some prime numbers $q_j, q_{j+k},$ where, $j, j+k \in \{1, 2, 3, \ldots, \sigma\}$. Then, the groups $C_{q_j^{\lambda_j}} \times C_{q_j^{\lambda_{j+k}}}$ and $C'_{q_j^{\lambda_j}} \times C'_{q_j^{\lambda_{j+k}}}$ are Sylow q_j subgroups of group A. With an argument similar to the previous one we can show that $C_{q_j^{\lambda_j}} \times C_{q_j^{\lambda_{j+k}}} = C'_{q_j^{\lambda_j}} \times C'_{q_j^{\lambda_{j+k}}}$. The last equation holds for *every* prime number q_j, $j \in \{1, 2, 3, \ldots, \sigma\}$, which appears twice in (25) (and in (27)). Therefore, we conclude that, in every case, $A = C_{q_1^{\lambda_1}} \times C_{q_2^{\lambda_2}} \times C_{q_3^{\lambda_3}} \times \cdots \times C_{q_\sigma^{\lambda_\sigma}} = C'_{q_1^{\lambda_1}} \times C'_{q_2^{\lambda_2}} \times C'_{q_3^{\lambda_3}} \times \cdots \times C'_{q_\sigma^{\lambda_\sigma}}$, and consequently expression (25) is unique for every subgroup of $C_n \times C_m$. This completes the proof. □

It is worth noting that the determination of the subgroups of $C_n \times C_m$ involves the prime decomposition of n and m. An attempt to find the subgroups of $C_n \times C_m$ without invoking the prime decomposition of n and m failed.

4. The cyclic subgroups of $C_{p^a} \times C_{p^\beta}$

In section 3 we showed that the problem of constructing the subgroups of $C_n \times C_m$ is reduced to the problem of constructing the subgroups of $C_{p^a} \times C_{p^\beta}$, where p is a prime number and a, β are non-negative integers. The group $C_{p^a} \times C_{p^\beta}$ is a finite Abelian group, and therefore, its rank is higher than the rank of any of its subgroups. Consequently, since $C_{p^a} \times C_{p^\beta}$ has two generators, its subgroups have either one or two generators. The following two propositions settle the question of determining the cyclic subgroups of $C_{p^a} \times C_{p^\beta}$. Proposition 2 refers to the case $1 \leqslant k \leqslant \min(a, \beta)$ and proposition 3 refers to the case $a < k \leqslant \beta$.

Proposition 2. *Let p be a prime number and let a and β be positive integers. Let C_{p^a} and C_{p^β} be cyclic groups of order p^a and p^β respectively. When $1 \leqslant k \leqslant \min(a, \beta)$ the direct product $C_{p^a} \times C_{p^\beta}$ has $p^k + p^{k-1}$ cyclic subgroups of order p^k. The generators of these subgroups are given by*

(i)

$$\left(x^{rp^{a-k}}, y^{p^{\beta-k}}\right), \qquad r \in \{0, 1, 2, \ldots, p^k - 1\}, \tag{28}$$

and,

(ii)

$$\left(x^{p^{a-k}}, y^{\rho p^{\beta-k+1}}\right), \qquad \rho \in \{0, 1, \ldots, p^{k-1} - 1\}, \tag{29}$$

where x and y are generators of the groups C_{p^a} and C_{p^β} respectively. The parameter r, which takes values in the set $\{0, 1, \ldots, p^k - 1\}$, and ρ, which takes values in the set $\{0, p, 2p, \ldots, (p^{k-1} - 1)p\}$, parameterize the distinct $p^k + p^{k-1}$ groups .

Proof. Let C_{p^a} and C_{p^β} be the cyclic groups of order p^a and p^β respectively, and let $C_{p^a} \times C_{p^\beta}$ be their direct product. In any cyclic group C_{p^k} there are $p^k - p^{k-1}$ elements of order p^k which generate the whole group. Therefore, when $1 \leqslant k \leqslant \min(a, \beta)$, the direct product $C_{p^a} \times C_{p^\beta}$ has $\frac{p^{2k} - p^{2(k-1)}}{p^k - p^{(k-1)}} = p^k + p^{k-1}$ cyclic subgroups of order p^k. Now that we have counted them, we proceed to construct them explicitly. Let x and y be generators of C_{p^a} and C_{p^β} respectively, and let (x^i, y^j) be an element of $C_{p^a} \times C_{p^\beta}$ which generates a subgroup of $C_{p^a} \times C_{p^\beta}$ isomorphic to C_{p^k}. Then i is divisible by p^{a-k} and j is divisible by $p^{\beta-k}$, and at least one of i, j is not divisible by a larger power of p. We distinguish the two cases.

(1) Either i is not divisible by p^{a-k+1}, or j is not divisible by $p^{\beta-k+1}$. Suppose first that j is *not* divisible by $p^{\beta-k+1}$. Since j is not divisible by $p^{\beta-k+1}$ we can write $j = p^{\beta-k} j_1$, where j_1 is not divisible by p. Since j_1 is not divisible by p there is a unique integer $u \in \{1, 2, \ldots, p^k - 1\}$ which solves the equation $j_1 u = 1 (\bmod p^k)$. Therefore, by multiplying both sides of $j = p^{\beta-k} j_1$ by u we obtain $uj = p^{\beta-k} (\bmod p^\beta)$. Since p does not divide u the elements (x^i, y^j) and (x^{ui}, y^{uj}) generate the same cyclic group C_{p^k}. Taking into account that $uj = p^{\beta-k} (\bmod p^\beta)$ the generator (x^{ui}, y^{uj}) is written as follows: $(x^{ui}, y^{uj}) = (x^{ui}, y^{p^{\beta-k}})$. Moreover, ui is a multiple of p^{a-k} and $ui = rp^{a-k} (\bmod p^a)$ for an $r \in \{0, 1, 2, \ldots, p^k - 1\}$. Therefore, in this case the cyclic groups C_{p^k} are generated by the elements

$$\left(x^{rp^{a-k}}, y^{p^{\beta-k}}\right), \qquad r \in \{0, 1, 2, \ldots, p^k - 1\}.$$

These generators generate different cyclic groups C_{p^k}.

(2) Next suppose that i is not divisible by p^{a-k+1} but j is divisible by $p^{\beta-k+1}$. With a method similar to that used in the previous case, one can show that in this case the cyclic groups C_{p^k} are generated by the elements

$$\left(x^{p^{a-k}}, y^{\rho p^{\beta-k+1}}\right), \qquad \rho \in \{0, 1, \ldots, p^{k-1} - 1\}.$$

These generators generate different cyclic groups C_{p^k}. These cyclic groups are different from those which were found in the first case. Now we have the correct number of subgroups, so clearly we have everything. This completes the proof. □

Proposition 3. *Let p be a prime number and let a be a non-negative integer and let β be a positive integer. Let C_{p^a} and C_{p^β} be cyclic groups of order p^a and p^β respectively. The direct product $C_{p^a} \times C_{p^\beta}$ has p^a cyclic subgroups of order p^k, where $a < k \leqslant \beta$. The generators of these subgroups are as follows,*

$$(x^j, y^{\beta-k}), \qquad j \in \{0, 1, 2, \ldots, p^a - 1\}, \tag{30}$$

where x and y are generators of the groups C_{p^a} and C_{p^β} respectively. The parameter j, which takes values in the set $\{0, 1, \ldots, p^a - 1\}$, parameterizes the groups.

Proof. We must have $k \leqslant \beta$ or there are no such cyclic subgroups at all. Since $a < k \leqslant \beta$ there are $p^a p^k - p^a p^{k-1}$ elements of $C_{p^a} \times C_{p^\beta}$ of order equal to p^k. In a cyclic group of order p^k there are $p^k - p^{k-1}$ elements of order p^k. Therefore, there are $\frac{p^a(p^k - p^{k-1})}{p^k - p^{k-1}} = p^a$

cyclic subgroups of $C_{p^a} \times C_{p^\beta}$ of order p^k. Evidently, the p^a subgroups of order p^k generated by $(1, y^{\beta-k})$, $(x, y^{\beta-k})$, $(x^2, y^{\beta-k})$, $\ldots$, $\left(x^{(p^a-1)}, y^{\beta-k}\right)$, where x is a generator of C_{p^a} and y is a generator of C_{p^β}, are different from one another. Henceforth, the p^a cyclic subgroups of $C_{p^a} \times C_{p^\beta}$ of order p^k, where $a < k \leqslant \beta$, are given by (30). A similar result holds when $a > k \geqslant \beta$. This completes the proof. □

5. Counting the subgroups with two generators of $C_{p^a} \times C_{p^\beta}$

Before constructing explicitly the subgroups of $C_{p^a} \times C_{p^\beta}$ which have two generators we firstly count them. The counting is an indispensable part of the construction: when we obtain the generators of the non-cyclic subgroups we firstly check that the generators are different and then we count them. If their number equals the number of subgroups we obtain in this section then we know that we have an exhaustive list of the subgroups. We applied the same method in section 4 when we found the generators of the cyclic subgroups of $C_{p^a} \times C_{p^\beta}$. However, it turns out that the counting of the non-cyclic subgroups is more involved than the counting of the cyclic ones. The method of counting presented here was suggested to me by Professor Charles Green.

Both the counting of the non-cyclic subgroups of $C_{p^a} \times C_{p^\beta}$ and their explicit construction in section 6 are based both on the second isomorphism theorem (see, for example, [21], p 104) and on the following lemma. The second isomorphism theorem (SIT) reads

Second isomorphism theorem. *Let N be a normal subgroup of the group G. Then there is a one-to-one correspondence between subgroups of G containing N and subgroups of G/N; and in it normal subgroups correspond to normal subgroups.*

We now state and prove the aforementioned lemma:

Lemma 1. *Let a, β be positive integers which satisfy $a \leqslant \beta$, let p be a prime number and let $C_{p^a} \times C_{p^\beta}$ be the direct product of the cyclic groups C_{p^a} and C_{p^β}. Let k be a positive integer such that $1 \leqslant k \leqslant a$. Then, the direct product $C_{p^a} \times C_{p^\beta}$ contains a unique copy of $C_{p^k} \times C_{p^k}$. Let l be a positive integer which satisfies $k \leqslant l \leqslant \beta$. Then, every subgroup $C_{p^k} \times C_{p^l}$ of $C_{p^a} \times C_{p^\beta}$ contains the unique copy $C_{p^k} \times C_{p^k}$.*

Proof. Let K be the unique subset of $G = C_{p^a} \times C_{p^\beta}$ which is defined by

$$K = \left\{g \in G : g^{p^k} = (I, I)\right\}, \tag{31}$$

where (I, I) is the identity element of G. The set K forms a group. Moreover, the order of K, $|K|$, is equal to p^{2k}. Indeed, if (k_1, k_2) is an element of K then $(k_1, k_2)^{p^k} = \left(k_1^{p^k}, k_2^{p^k}\right) = (I, I)$,, and so, since $k \leqslant a \leqslant \beta$, there are p^k possibilities for k_1 and p^k possibilities for k_2. Therefore, in total, there are $p^k p^k = p^{2k}$ possibilities and consequently K has p^{2k} elements. Let H be a subgroup of G which is isomorphic to $C_{p^k} \times C_{p^k}$. If $g_H = (h_1, h_2)$ is an element of H then $h_1^{p^k} = h_2^{p^k} = I$. Therefore $(h_1, h_2) \in K$ and so we conclude that H is a subgroup of K. But the order of H, $|H| = |K| = p^{2k}$ and henceforth $H = K$. Consequently, there is only one subgroup of $C_{p^a} \times C_{p^\beta}$ which is isomorphic to $C_{p^k} \times C_{p^k}$.

Let $C_{p^k} \times C_{p^l}$ denote a subgroup of $C_{p^a} \times C_{p^\beta}$ which is isomorphic to the direct product of cyclic groups C_{p^k} and C_{p^l}, where $k \leqslant a$ and $k \leqslant l \leqslant \beta$. Let π_i, $i = 1, 2$, be the projections onto the factors of $C_{p^k} \times C_{p^l}$. There is a unique copy $\mathcal{A}$ of C_{p^k} in $\pi_1(C_{p^k} \times C_{p^l})$ and a unique copy $\mathcal{B}$ of C_{p^k} in $\pi_2(C_{p^k} \times C_{p^l})$. Therefore $\mathcal{A} \times \mathcal{B}$ is isomorphic to $C_{p^k} \times C_{p^k}$ and is a subgroup of $C_{p^k} \times C_{p^l}$. But there is only one copy of $C_{p^k} \times C_{p^k}$ in $C_{p^a} \times C_{p^\beta}$. Henceforth $\mathcal{A} \times \mathcal{B}$ is identical to this unique copy and $C_{p^k} \times C_{p^l}$ contains it. This completes the proof. □

By using the SIT and the previous lemma we can now calculate the number of the non-cyclic subgroups which are contained in $C_{p^a} \times C_{p^\beta}$. This is the content of the following proposition.

Proposition 4. *Let a, β be positive integers which satisfy $a \leqslant \beta$, let p be a prime number and let $G = C_{p^a} \times C_{p^\beta}$ be the direct product of the cyclic groups C_{p^a} and C_{p^β}. Let k be a non-negative integer and let l be a positive integer such that $0 \leqslant k \leqslant a$ and $k < l$. Then we have the following:*

(i) When $0 \leqslant k < l \leqslant a$ the group G contains $p^{l-k} + p^{l-k-1}$ copies of $C_{p^k} \times C_{p^l}$.
(ii) When $0 \leqslant k \leqslant a < l \leqslant \beta$ the group G contains p^{a-k} copies of $C_{p^k} \times C_{p^l}$.

Proof. When $k = 0$ and $0 < l \leqslant a$ proposition 2 shows that there are $p^l + p^{l-1}$ cyclic subgroups C_{p^l} of the group G. When $k = 0$ and $0 \leqslant a < l \leqslant \beta$ proposition 3 shows that there are p^a cyclic subgroups C_{p^l} of the group G. This proves the proposition when $k = 0$. So now it can be assumed that $k > 0$. Fix integers k and l such that $0 < k < l \leqslant \beta$. Any subgroup H of G which is isomorphic to $C_{p^k} \times C_{p^l}$ contains the unique copy $N = C_{p^k} \times C_{p^k}$. According to SIT the subgroups H are in one-to-one correspondence with the subgroups H/N of G/N. We have $H/N \cong C_{p^{l-k}}$ and $G/N \cong C_{p^{a-k}} \times C_{p^{\beta-k}}$. So, the problem is reduced to counting the cyclic subgroups $C_{p^{l-k}}$ of the direct product $C_{p^{a-k}} \times C_{p^{\beta-k}}$. The answer to this question is given by propositions 2 and 3. We distinguish the two cases. (1) When $0 < l - k \leqslant a - k \leqslant \beta - k$ according to proposition 2 there are $p^{l-k} + p^{l-k-1}$ cyclic subgroups $C_{p^{l-k}}$ of the direct product $C_{p^{a-k}} \times C_{p^{\beta-k}}$. From SIT it follows that the group G has $p^{l-k} + p^{l-k-1}$ subgroups isomorphic to $C_{p^k} \times C_{p^l}$. (2) When $0 \leqslant a - k < l - k \leqslant \beta - k$ according to proposition 3 there are p^{a-k} cyclic subgroups $C_{p^{l-k}}$ of the group $C_{p^{a-k}} \times C_{p^{\beta-k}}$. From SIT it follows that the group G has p^{a-k} subgroups isomorphic to $C_{p^k} \times C_{p^l}$. This completes the proof. $\square$

6. The subgroups of $C_{p^a} \times C_{p^\beta}$ with two generators

In section 4 we found the cyclic subgroups of $C_{p^a} \times C_{p^\beta}$. We proceed now to determine the generators of the non-cyclic subgroups of $C_{p^a} \times C_{p^\beta}$, i.e., of those subgroups which have two generators. There are many ways to accomplish this. The one presented here was suggested to me by Professor Peter Cameron, and it seems to be the shorter and neater one.

6.1. Description of the method

We construct explicitly the subgroups $C_{p^k} \times C_{p^l}$ of $C_{p^a} \times C_{p^\beta}$ when $0 < k < l \leqslant \beta$ and $a \leqslant \beta$. Let H be a group isomorphic to $C_{p^k} \times C_{p^l}$ and let N denote the unique copy $C_{p^k} \times C_{p^k}$. Let G denote the direct product $C_{p^a} \times C_{p^\beta}$ and let x be a generator of the group C_{p^a} and let y be a generator of the group C_{p^β}. Let $(h_1, h_2)N$ be a generator of the cyclic group H/N. The elements of the group H are generated by the three elements (h_1, h_2), $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$. Since the group H has two generators any one of the elements (h_1, h_2), $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$ can be expressed via the other two. The two independent elements so obtained generate all the elements of group H. So, the problem of finding, for fixed k and l, all the subgroups $C_{p^k} \times C_{p^l}$ of $C_{p^a} \times C_{p^\beta}$, when $0 < k < l \leqslant \beta$ and $a \leqslant \beta$, is reduced to finding the answers to the following two questions:

(i) Find the generators $(h_1, h_2)N$ of all cyclic subgroups $H/N \cong C_{p^{l-k}}$ of the direct product $G/N \cong C_{p^{a-k}} \times C_{p^{\beta-k}}$.

(ii) From each element (h_1, h_2) obtained by answering question 1 and from the generators $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$ of N construct two independent generators of each group H.

According to SIT the list of groups H so obtained is exhaustive.

6.2. The non-cyclic subgroups of $C_{p^a} \times C_{p^\beta}$

We distinguish the two cases:

(1) $0 < k < l \leqslant a$: therefore, in this case we have $0 < l - k \leqslant a - k \leqslant \beta - k$. In the previous subsection it was shown that the problem of finding , for fixed k and l, all the subgroups $C_{p^k} \times C_{p^l}$ of $C_{p^a} \times C_{p^\beta}$ is reduced to answering two questions. Firstly, we answer question 1.

(1) A generator of the subgroup $(C_{p^{a-k}}, I) \cong C_{p^{a-k}}$ of $G/N \cong C_{p^{a-k}} \times C_{p^{\beta-k}}$ is the element $(x, I)N$ of the group G/N. Similarly, the element $(I, y)N$ is a generator of the group $(I, C_{p^{\beta-k}}) \cong C_{p^{\beta-k}}$. From proposition 2 we conclude that the direct product $G/N \cong C_{p^{a-k}} \times C_{p^{\beta-k}}$ has $p^{l-k} + p^{l-k-1}$ cyclic subgroups isomorphic to $H/N \cong C_{p^{l-k}}$. From these $p^{l-k} + p^{l-k-1}$ cyclic subgroups p^{l-k} are generated by the generators

$$((x, I)N)^{rp^{(a-k)-(l-k)}}((I, y)N)^{p^{(\beta-k)-(l-k)}} = \left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right)N, \qquad r \in \{0, 1, 2, \ldots, p^{l-k} - 1\}, \tag{32}$$

and the rest p^{l-k-1} cyclic subgroups of $C_{p^{a-k}} \times C_{p^{\beta-k}}$ which are isomorphic to $C_{p^{l-k}}$ are generated by the elements

$$((x, I)N)^{p^{(a-k)-(l-k)}}((I, y)N)^{\rho p^{(\beta-k)-(l-k)+1}} = \left(x^{p^{a-l}}, y^{\rho p^{\beta-l+1}}\right)N, \tag{33}$$

where, $\rho \in \{0, 1, 2, \ldots, p^{l-k-1} - 1\}$. Now we proceed with answering question 2.

(2) When $0 < k < l \leqslant a \leqslant \beta$ the elements of the subgroups $H \cong C_{p^k} \times C_{p^l}$ of the group $G = C_{p^a} \times C_{p^\beta}$ are generated either by the three elements $\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right)$, $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$, where $r \in \{0, 1, 2, \ldots, p^{l-k} - 1\}$, or by the three elements $\left(x^{p^{a-l}}, y^{\rho p^{\beta-l+1}}\right)$, $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$, where $\rho \in \{0, 1, 2, \ldots, p^{l-k-1} - 1\}$. We examine in turn now these two cases.

(2a) Let H be any one of the p^{l-k} subgroups of $C_{p^a} \times C_{p^\beta}$ which is isomorphic to $C_{p^k} \times C_{p^l}$ and which is generated by the three elements $\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right)$, $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$, where $r \in \{0, 1, 2, \ldots, p^{l-k} - 1\}$. The group H has two independent generators and therefore any one of the three elements $\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right)$, $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$ can be generated by the other two. In fact, if we raise the element $\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right)$ to the power p^{l-k} we obtain $\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right)^{p^{l-k}} = \left(x^{rp^{a-k}}, y^{p^{\beta-k}}\right)$. By multiplying $\left(x^{rp^{a-k}}, y^{p^{\beta-k}}\right)$ by the inverse of $\left(x^{rp^{a-k}}, I\right)$ we obtain $\left(x^{rp^{a-k}}, y^{p^{\beta-k}}\right)\left(\left(x^{rp^{a-k}}\right)^{-1}, I\right) = \left(I, y^{p^{\beta-k}}\right)$. Therefore, we can discard the generator $\left(I, y^{p^{\beta-k}}\right)$. We conclude, that the p^{l-k} subgroups $H \cong C_{p^k} \times C_{p^l}$ are generated by the elements

$$\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right), \left(x^{p^{a-k}}, I\right), \qquad \text{where} \quad r \in \{0, 1, 2, \ldots, p^{l-k} - 1\}. \tag{34}$$

(2b) Let H be now any one of the p^{l-k-1} subgroups of $C_{p^a} \times C_{p^\beta}$ which is isomorphic to $C_{p^k} \times C_{p^l}$ and which is generated by the three elements $\left(x^{p^{a-l}}, y^{\rho p^{\beta-l+1}}\right)$, $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$, where $\rho \in \{0, 1, 2, \ldots, p^{l-k-1} - 1\}$. It can be shown as before that the generator $\left(x^{p^{a-k}}, I\right)$ can be discarded. We conclude that the p^{l-k-1} subgroups $H \cong C_{p^k} \times C_{p^l}$ are generated by the elements

$$\left(x^{p^{a-l}}, y^{\rho p^{\beta-l+1}}\right), \left(I, y^{p^{\beta-k}}\right), \qquad \text{where} \quad \rho \in \{0, 1, 2, \ldots, p^{l-k-1} - 1\}. \tag{35}$$

The number of the subgroups which are given in (34) and (35) coincides with the number given in proposition 4.

(2) $0 < k \leqslant a < l \leqslant \beta$: therefore, we have $0 \leqslant a - k < l - k \leqslant \beta - k$. It is to be noted that in this case a cannot be equal to β and is strictly smaller than β. In section 6.1 it was shown that the problem of finding, for fixed k and l, all the subgroups $C_{p^k} \times C_{p^l}$ of $C_{p^a} \times C_{p^\beta}$ is reduced to answering two questions. Firstly, we answer question 1.

(1) From proposition 3 we conclude that the direct product $G/N \cong C_{p^{a-k}} \times C_{p^{\beta-k}}$ has p^{a-k} cyclic subgroups isomorphic to $H/N \cong C_{p^{l-k}}$. These p^{a-k} cyclic subgroups are generated by the elements

$$((x, I)N)^j ((I, y)N)^{p^{(\beta-k)-(l-k)}} = \left(x^j, y^{p^{\beta-l}}\right)N, \qquad j \in \{0, 1, 2, \ldots, p^{a-k} - 1\}. \tag{36}$$

Now we proceed with answering question 2.

(2) As it was explained in section 6.1 when $0 < k \leqslant a < l \leqslant \beta$ the elements of all the subgroups $H \cong C_{p^k} \times C_{p^l}$ of the group $G = C_{p^a} \times C_{p^\beta}$ are generated by the three elements $\left(x^j, y^{p^{\beta-l}}\right)$, $\left(x^{p^{a-k}}, I\right)$ and $\left(I, y^{p^{\beta-k}}\right)$, where $j \in \{0, 1, 2, \ldots, p^{a-k} - 1\}$. We can prove as before that the generator $\left(I, y^{p^{\beta-k}}\right)$ can be discarded. It is concluded that the p^{a-k} subgroups $H \cong C_{p^k} \times C_{p^l}$ are generated by the elements

$$\left(x^j, y^{p^{\beta-l}}\right), \left(x^{p^{a-k}}, I\right), \qquad \text{where} \quad j \in \{0, 1, 2, \ldots, p^{a-k} - 1\}. \tag{37}$$

The number of subgroups given in (37) equals to the number given in proposition 4. So now we have all the subgroups $C_{p^k} \times C_{p^l}$ of $C_{p^a} \times C_{p^\beta}$. We summarize the previous results in the following theorem.

Theorem 1. *Let p be a prime number and let k, l, a, β be integers which satisfy $0 < k < l \leqslant \beta$ and $a \leqslant \beta$. Let $C_{p^a} \times C_{p^\beta}$ denote the direct product of the cyclic groups C_{p^a} and C_{p^β} and let $C_{p^k} \times C_{p^l}$ denote the direct product of the cyclic groups C_{p^k} and C_{p^l}. Then we have the following:*

(i) When $0 < k < l \leqslant a \leqslant \beta$ the group $C_{p^a} \times C_{p^\beta}$ has $p^{l-k} + p^{l-k-1}$ subgroups which are isomorphic to the group $C_{p^k} \times C_{p^l}$. From these subgroups, p^{l-k} are generated by the elements

$$\left(x^{rp^{a-l}}, y^{p^{\beta-l}}\right), \left(x^{p^{a-k}}, I\right), \qquad \text{where} \quad r \in \{0, 1, 2, \ldots, p^{l-k} - 1\},$$

and the remaining p^{l-k-1} subgroups are generated by the elements

$$\left(x^{p^{a-l}}, y^{\rho p^{\beta-l+1}}\right), \left(I, y^{p^{\beta-k}}\right), \qquad \text{where} \quad \rho \in \{0, 1, 2, \ldots, p^{l-k-1} - 1\}.$$

(ii) When $0 < k \leqslant a < l \leqslant \beta$ the group $C_{p^a} \times C_{p^\beta}$ has p^{a-k} subgroups which are isomorphic to the group $C_{p^k} \times C_{p^l}$. These p^{a-k} subgroups are generated by the elements

$$\left(x^j, y^{p^{\beta-l}}\right), \left(x^{p^{a-k}}, I\right), \qquad \text{where} \quad j \in \{0, 1, 2, \ldots, p^{a-k} - 1\}.$$

This completes our consideration of the non-cyclic subgroups $C_{p^k} \times C_{p^l}$ of the group $C_{p^a} \times C_{p^\beta}$.

7. Generators of the subgroups of $C_n \times C_m$

The group $C_n \times C_m$ is a finite Abelian group, and therefore, its rank is higher than the rank of any of its subgroups. Consequently, since $C_n \times C_m$ has two generators, its subgroups have either one or two generators. In this section, and in particular in corollary 1, and in theorem 2, prime decomposition of n and m is employed, i.e., it is assumed that $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ and $m = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}$, where $p_i, i = 1, 2, \ldots, s$, are distinct prime numbers and a_i, β_i are non-negative integers. Then we can write (equation (26)) $C_n \times C_m = \left(C_{p_1^{a_1}} \times C_{p_1^{\beta_1}}\right) \times \left(C_{p_2^{a_2}} \times C_{p_2^{\beta_2}}\right) \times \left(C_{p_3^{a_3}} \times C_{p_3^{\beta_3}}\right) \times \cdots \times \left(C_{p_s^{a_s}} \times C_{p_s^{\beta_s}}\right)$. It is also assumed that $p_1 < p_2 < \cdots < p_s$. The structure of the subgroups of $C_n \times C_m$ is deduced from proposition 1. Indeed, if in equation (25) each q_i, $i = 1, 2, \ldots, \sigma$, occurs only once then a cyclic subgroup of $C_n \times C_m$ is obtained. If at least one of the q_i, $i = 1, 2, \ldots, \sigma$, occurs twice then a non-cyclic subgroup of $C_n \times C_m$ results. Using the previous observation, we can formulate the general form of the subgroups of $C_n \times C_m$. This is the content of the following corollary which is an immediate consequence of proposition 1.

Corollary 1. *Any subgroup $\mathcal{C}$ of $C_n \times C_m$ can be written as a direct product*

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \cdots \times A_s. \tag{38}$$

If $\mathcal{C}$ has one generator then A_i is a cyclic subgroup of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i = 1, 2, \ldots, s$ and vice versa. If $\mathcal{C}$ has two generators then at least one A_i is a non-cyclic subgroup of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i = 1, 2, \ldots, s$, and vice versa. For every subgroup $\mathcal{C}$ of $C_n \times C_m$ the expression (38) is unique.

The knowledge of the general form of the subgroups of $C_n \times C_m$ does not suffice for the construction of the IRs of $B(2, 2)$. We actually need the explicit form of the subgroups of $C_n \times C_m$, i.e., we need their generators. For this purpose, it will be convenient to employ here the following [23]:

Lemma 2. *Let s be a positive integer and let $p_1, p_2, \ldots, p_s$ be distinct prime numbers. Let G be the direct product of cyclic groups*

$$G = C_{p_1^{k_1}} \times C_{p_2^{k_2}} \times \cdots \times C_{p_s^{k_s}}, \tag{39}$$

where $k_1, k_2, \ldots, k_s$ are positive numbers. Let g_i be a generator of the cyclic group $C_{p_i^{k_i}}$, $i = 1, 2, \ldots, s$. Then the element

$$g = g_1 \cdot g_2 \cdots g_s, \tag{40}$$

where $\cdot$ denotes the group composition law in the group G, is a generator of the cyclic group G.

The writing of explicit expressions of the generators of the subgroups of $C_n \times C_m$ is facilitated by the use of the set $S_{\mathcal{P}}$ of the permutations of s pairs of numbers $\left(p_1^{a_1}, p_1^{\beta_1}\right), \left(p_2^{a_2}, p_2^{\beta_2}\right), \ldots, \left(p_s^{a_s}, p_s^{\beta_s}\right)$. Let $\mathcal{P} \in S_{\mathcal{P}}$. By

$$\mathcal{P}\left(p_i^{a_i}, p_i^{\beta_i}\right) = \left(p_j^{a_j}, p_j^{b_j}\right), \qquad i, j = 1, 2, \ldots, s, \tag{41}$$

we denote that the permutation $\mathcal{P}$ moves the pair $\left(p_i^{a_i}, p_i^{\beta_i}\right)$ at the ith position, to the jth position. Obviously, $p_i \equiv p_j, a_i \equiv a_j$, and $\beta_i \equiv b_j$. By using theorem 1, propositions 2 and 3, corollary 1, and lemma 2 we can give explicit expressions for the generators of the subgroups of $C_n \times C_m$. This is the content of the following theorem.

Theorem 2. *Let $\mathcal{C}$ be a subgroup of $\mathcal{C}$ of $C_n \times C_m$. We distinguish the two cases:*

(i) When $\mathcal{C}$ is non-cyclic then it can be written in a highly non-unique way as a direct product of two cyclic groups $\mathcal{C}_1$ and *$\mathcal{C}_2$, i.e.,*

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2, \tag{42}$$

whose orders are not relatively prime. A legitimate choice for the generators g_1 and g_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the following:

$$g_1 = \left(x^{\mathcal{A}_1}, y^{\mathcal{B}_1}\right), \tag{43}$$

where x is a generator of C_n, y is a generator of C_m,

$$\frac{\mathcal{A}_1}{n} = \sum_{i=1}^{\nu} r_i p_i^{-k_i} + \sum_{i=\nu+1}^{\nu+\chi} p_i^{-k_i} + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} j_i / p_i^{a_i} + \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} p_i^{-k_i} + \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} r_i p_i^{-k_i}$$
$$+ \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} p_i^{-k_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} t_i / p_i^{a_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{-k_i}, \tag{44}$$

and

$$\frac{\mathcal{B}_1}{m} = \sum_{i=1}^{\nu} p_i^{-k_i} + \sum_{i=\nu+1}^{\nu+\chi} \rho_i p_i^{-k_i+1} + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} p_i^{-k_i} + \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} j_i / p_i^{b_i} + \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} p_i^{-k_i}$$
$$+ \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} \rho_i p_i^{-k_i+1} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{-k_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} t_i / p_i^{b_i}, \tag{45}$$

$$g_2 = (x^{\mathcal{A}_2}, y^{\mathcal{B}_2}), \tag{46}$$

where

$$\frac{\mathcal{A}_2}{n} = \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} p_i^{-l_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{-l_i} \tag{47}$$

and

$$\frac{\mathcal{B}_2}{m} = \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} p_i^{-l_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{-l_i}. \tag{48}$$

(ii) When $\mathcal{C}$ is cyclic then it is generated by

$$g = (x^{\mathcal{A}}, y^{\mathcal{B}}), \tag{49}$$

where

$$\mathcal{A} = \mathcal{A}_1 \qquad \text{and} \qquad \mathcal{B} = \mathcal{B}_1 \qquad \text{when} \quad \sigma = \theta = \phi = \xi = 0. \tag{50}$$

The non-negative integers $\nu, \chi, \tau, \psi, \sigma, \theta, \phi, \xi$ are such that $\nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi \leqslant s$. When $\mathcal{C}$ is cyclic then $\sigma = \theta = \phi = \xi = 0$. When $\mathcal{C}$ is non-cyclic then at least one of the $\sigma, \theta, \phi, \xi$ must be non-zero. Moreover, $\left(p_j^{a_j}, p_j^{b_j}\right) = \mathcal{P}\left(p_i^{a_i}, p_i^{\beta_i}\right)$, $i, j = 1, \ldots, s$, for some permutation $\mathcal{P}$ of the s pairs of numbers $\left(p_1^{a_1}, p_1^{\beta_1}\right), \left(p_2^{a_2}, p_2^{\beta_2}\right), \ldots, \left(p_s^{a_s}, p_s^{\beta_s}\right)$. Furthermore, the allowed values of the other indices are easily deduced from propositions 2, 3 and theorem 1.

Proof. By prime decomposing n and m we obtain equation (26). According to corollary 1 every subgroup $\mathcal{C}$ of $C_n \times C_m$ has the form $\mathcal{C} = A_1 \times A_2 \times A_3 \times \cdots \times A_s$, where, when $\mathcal{C}$ is cyclic, then, A_i is a cyclic subgroup of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i = 1, 2, \ldots, s$, and when $\mathcal{C}$ is non-cyclic at least one A_i is a non-cyclic subgroup of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i = 1, 2, \ldots, s$. The possible choices for the groups A_i, $i = 1, 2, \ldots, s$, are given by propositions 2 and 3 and theorem 1. Consider first the case of a non-cyclic $\mathcal{C}$. We have to realize all possibilities for $\mathcal{C}$. To this end, we choose the A_i, $i = 1, 2, \ldots, s$, groups as follows: ν to be of the form (28), χ to be of the form (29), $\tau + \psi$ to be of the form (30), σ to be of the form (34), θ to be of the form (35), and finally,

$\phi + \xi$ to be of the form (37). Obviously, $\nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi \leqslant s$. To allow for at least one A_i to be non-cyclic at least one of $\sigma, \theta, \phi, \xi$ must be non-zero. Having the choice for the groups A_i specified, still there is a great deal of freedom in writing $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, i.e., as a direct product of two cyclic groups $\mathcal{C}_1$ and $\mathcal{C}_2$. Let x be a generator of C_n and y be a generator of C_m. By using lemma 2 we find that a possible choice for the generators g_1 and g_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, is given by

$$
\begin{aligned}
g_1 = \prod_{i=1}^{\nu} \left(x_i^{r_i p_i^{p_i^{a_i-k_i}}}, y_i^{p_i^{p_i^{b_i-k_i}}} \right) \cdot \prod_{i=\nu+1}^{\nu+\chi} \left(x_i^{p_i^{p_i^{a_i-k_i}}}, y_i^{\rho_i p_i^{p_i^{b_i-k_i+1}}} \right) \cdot \prod_{i=\nu+\chi+1}^{\nu+\chi+\tau} \left(x_i^{j_i}, y_i^{p_i^{p_i^{b_i-k_i}}} \right) \\
\cdot \prod_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} \left(x_i^{p_i^{p_i^{a_i-k_i}}}, y_i^{j_i} \right) \cdot \prod_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} \left(x_i^{r_i p_i^{p_i^{a_i-k_i}}}, y_i^{p_i^{p_i^{b_i-k_i}}} \right) \\
\cdot \prod_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} \left(x_i^{p_i^{p_i^{a_i-k_i}}}, y_i^{\rho_i p_i^{p_i^{b_i-k_i+1}}} \right) \\
\cdot \prod_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} \left(x_i^{t_i}, y_i^{p_i^{p_i^{b_i-k_i}}} \right) \cdot \prod_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} \left(x_i^{p_i^{p_i^{a_i-k_i}}}, y_i^{t_i} \right),
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
g_2 = \prod_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} \left(x_i^{p_i^{p_i^{a_i-l_i}}}, I \right) \cdot \prod_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} \left(x_i^{p_i^{p_i^{a_i-l_i}}}, I \right) \cdot \prod_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} \left(I, y_i^{p_i^{p_i^{b_i-l_i}}} \right) \\
\cdot \prod_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} \left(I, y_i^{p_i^{p_i^{b_i-l_i}}} \right),
\end{aligned}
\tag{52}
$$

where I is the identity element and $\prod$ denotes group multiplication in $C_n \times C_m$. To simplify notation in equations (51) and (52) we have set $x_i = x^{(n/p_i^{a_i})}$ and $y_i = y^{(m/p_i^{b_i})}$. To account for all possible choices for the groups A_i in (51) and (52) we have employed the elements of $S_{\mathcal{P}}$. In (51) it is assumed that $a_w < k_w \leqslant b_w, a_y \geqslant k_y > b_y$, where $w \in \{\nu + \chi + 1, \ldots, \nu + \chi + \tau\}$, and $y \in \{\nu + \chi + \tau + 1, \ldots, \nu + \chi + \tau + \psi\}$. Moreover, in (52) it is assumed that $1 \leqslant l_{w_1} \leqslant a_{w_1} < k_{w_1} \leqslant b_{w_1}$, and , $1 \leqslant l_{y_1} \leqslant b_{y_1} < k_{y_1} \leqslant a_{y_1}$, where $w_1 \in \{\nu + \chi + \tau + \psi + \sigma + \theta + 1, \ldots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi\}$, and $y_1 \in \{\nu + \chi + \tau + \psi + \sigma + \theta + \phi + 1, \ldots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi\}$. The values of the rest of the indices which appear (51) and (52) are easily deduced from propositions 2 , 3 and theorem 1. Addition of the exponents of x and y in (51) yields that $g_1 = (x^{\mathcal{A}_1}, y^{\mathcal{B}_1})$, where $\mathcal{A}_1$ and $\mathcal{B}_1$ are given by (44) and (45) correspondingly. Similarly, addition of the exponents of x and y in (52) gives that $g_2 = (x^{\mathcal{A}_2}, y^{\mathcal{B}_2})$, where $\mathcal{A}_2$ and $\mathcal{B}_2$ are given by (47) and (48) respectively. Consider now the case of a cyclic $\mathcal{C}$. Actualizing all possibilities for $\mathcal{C}$ necessitates that the following choice for the groups $A_i, i = 1, 2, \ldots, s$: ν must be of the form (28), χ must be of the form (29), and finally, $\tau + \psi$ must be of the form (30). By using lemma 2 we find that a generator g of $\mathcal{C}$ is given by equation (51) when $\sigma = \theta = \phi = \xi = 0$. Addition of the exponents of x and y in (51) gives that $g = (x^{\mathcal{A}}, y^{\mathcal{B}})$, where $\mathcal{A} = \mathcal{A}_1$ and $\mathcal{B} = \mathcal{B}_1$ when $\sigma = \theta = \phi = \xi = 0$. This completes the proof. $\square$

The order $|\mathcal{C}_1|$ of $\mathcal{C}_1$ is obtained from (43), (44) and (45), and similarly the order $|\mathcal{C}_2|$ of $\mathcal{C}_2$ results from (46), (47) and (48), and they are given by

$$
|\mathcal{C}_1| = \prod_{i=1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{k_i}, \qquad \text{and} \qquad |\mathcal{C}_2| = \prod_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{l_i}. \tag{53}
$$

For the purposes of representation theory it is convenient to rewrite the non-cyclic subgroups $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ as subgroups of $SO(2) \times SO(2)$:

$$\mathcal{C} = \left(R\left(\frac{2\pi}{n}(\mathcal{A}_1 i + \mathcal{A}_2 j)\right), R\left(\frac{2\pi}{m}(\mathcal{B}_1 i + \mathcal{B}_2 j)\right)\right), \tag{54}$$

where $i \in \{0, \ldots, |\mathcal{C}_1| - 1\}$ and $j \in \{0, \ldots, |\mathcal{C}_2| - 1\}$. For future reference we write explicitly the coefficients $\mathcal{A}$, $\mathcal{B}$ which appear in (49),

$$\frac{\mathcal{A}}{n} = \sum_{i=1}^{\nu} r_i p_i^{-k_i} + \sum_{i=\nu+1}^{\nu+\chi} p_i^{-k_i} + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} j_i \big/ p_i^{a_i} + \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} p_i^{-k_i} \tag{55}$$

and

$$\frac{\mathcal{B}}{m} = \sum_{i=1}^{\nu} p_i^{-k_i} + \sum_{i=\nu+1}^{\nu+\chi} \rho_i p_i^{-k_i+1} + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} p_i^{-k_i} + \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} j_i \big/ p_i^{b_i}. \tag{56}$$

A cyclic subgroup $\mathcal{C}$ of $C_n \times C_m$ can easily be rewritten as a subgroup of $SO(2) \times SO(2)$:

$$\mathcal{C} = \left(R\left(\left(\frac{2\pi}{n}\mathcal{A}\right) i\right), R\left(\left(\frac{2\pi}{m}\mathcal{B}\right) i\right)\right), \tag{57}$$

$i \in \{0, \ldots, |\mathcal{C}| - 1\}$. Finally, $|\mathcal{C}| = |\mathcal{C}_1|$ when $\sigma = \theta = \phi = \xi = 0$, i.e.,

$$|\mathcal{C}| = \prod_{i=1}^{\nu+\chi+\tau+\psi} p_i^{k_i}. \tag{58}$$

8. Explicit description of the finite little groups of $B(2, 2)$

The finite little groups $\mathcal{C}$ of $B(2,2)$ are those subgroups of $C_n \times C_m$ which contain the element $(-I, -I)$ [19], i.e., the group $Z_2 = \{(I, I), (-I, -I)\}$. Since $\pi_1(\mathcal{C}) = C_n$ and $\pi_2(\mathcal{C}) = C_m, \pi_1(\mathcal{C})$ and $\pi_2(\mathcal{C})$ must each contain the element $-I$, both n and m must be even. Therefore, in the prime decomposition of $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ and $m = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}$, $p_1 = 2$, and $a_1 \cdot \beta_1 \neq 0$ (it is assumed that $p_1 < p_2 < \cdots < p_s$). The group $Z_2 = \{(I, I), (-I, -I)\}$ can only be contained in the factor $C_{p_1^{a_1}} \times C_{p_1^{\beta_1}} = C_{2^{a_1}} \times C_{2^{\beta_1}}$ of

$$C_n \times C_m = \left(C_{p_1^{a_1}} \times C_{p_1^{\beta_1}}\right) \times \left(C_{p_2^{a_2}} \times C_{p_2^{\beta_2}}\right) \times \left(C_{p_3^{a_3}} \times C_{p_3^{\beta_3}}\right) \times \cdots \times \left(C_{p_s^{a_s}} \times C_{p_s^{\beta_s}}\right),$$

(equation (26)). Henceforth, $A - B$, where A and B are sets, signifies those elements of A which do not belong to B. The following two observations aim at determining the structure of the little groups of $B(2, 2)$. Firstly, from propositions 2 and 3, we observe that the *only* cyclic subgroups $C_{2^{k_1}}$ of $C_{2^{a_1}} \times C_{2^{\beta_1}}$ which contain the group Z_2 are generated by

$$\left(x^{r_1 2^{a_1-k_1}}, y^{2^{\beta_1-k_1}}\right), r_1 \in \{1, 2, \ldots, 2^{k_1} - 1\} - \{2, 2 \cdot 2, \ldots, (2^{k_1-1} - 1)2\}, \tag{59}$$

where x and y are generators of the groups $C_{2^{a_1}}$ and $C_{2^{\beta_1}}$ respectively, r_1 parameterizes the groups, and $1 \leqslant k_1 \leqslant \min(a_1, \beta_1)$. Therefore, *any* cyclic little group is obtained by restricting the group A_1 in equation (38) to be one of the groups whose generators are given in equation (59). We note that when $k_1 = 1$ we obtain $A_1 = Z_2$. Secondly, we observe that according to theorem 1 *all* the non-cyclic subgroups of $C_{2^{a_1}} \times C_{2^{\beta_1}}$ contain the group Z_2. In equation (26) Z_2 can now be contained in a cyclic or a non-cyclic subgroup of $C_{2^{a_1}} \times C_{2^{\beta_1}}$. Therefore, *any* non-cyclic little group is obtained by restricting the group A_1 in equation (38) to be one of the cyclic subgroups of $C_{2^{a_1}} \times C_{2^{\beta_1}}$ given in equation (59) or *any* of the non-cyclic subgroups of $C_{2^{a_1}} \times C_{2^{\beta_1}}$ given in theorem 1. We conclude that

Theorem 3. *The cyclic little groups of $B(2,2)$ are precisely those cyclic subgroups of $C_n \times C_m$ which contain one of the groups given in equation (59). The non-cyclic little groups of $B(2,2)$ are precisely those non-cyclic subgroups of $C_n \times C_m$ which contain either one of the groups given in equation (59) or any of the non-cyclic subgroups of $C_{2^{\alpha_1}} \times C_{2^{\beta_1}}$ given in theorem 1.*

However, what we actually need is explicit expressions for the generators of the little groups of $B(2,2)$. This is attained in the next two theorems which are readily obtained by combining the results of theorems 2 and 3 and which are given without proof. The first theorem describes the generators of the cyclic little groups of $B(2,2)$:

Theorem 4. *Let $\mathcal{C}$ be a cyclic little group $\mathcal{C}$ of $B(2,2)$. Then $\mathcal{C}$ is generated by (49) where n and m are positive even numbers. Furthermore, in (55) and (56) one of the primes $p_1, p_2, \ldots, p_\nu$ is the number 2. If, say, $p_t = 2, t \in \{1, 2, \ldots, \nu\}$ then $r_t \in \{1, 2, \ldots, 2^{k_t} - 1\} - \{2, 2 \cdot 2, \ldots, (2^{k_t-1} - 1)2\}$, and $a_t \cdot b_t \neq 0$. The other indices which appear in (55)and (56) take the values given at the end of theorem 2. The cyclic little group $\mathcal{C}$ can alternatively be written as in equation (57), where, in expressions (55) and (56) of $\mathcal{A}$ and $\mathcal{B}$ respectively, the aforementioned restrictions on the indices involved have been taken into account.*

The second theorem gives the generators of the non-cyclic little groups of $B(2,2)$:

Theorem 5. *Let $\mathcal{C}$ be a non-cyclic little group $\mathcal{C}$ of $B(2,2)$. A legitimate choice for the generators of $\mathcal{C}$ is given by (43) and (46), where n and m are positive even numbers. We distinguish the two cases:*

(i) The group Z_2 is contained in a cyclic subgroup of $C_{2^{a_t}} \times C_{2^{b_t}}$. In this case, in (44) and (45) one of the primes $p_1, p_2, \ldots, p_\nu$ equals to 2. If, say, $p_t = 2, t \in \{1, 2, \ldots, \nu\}$, then $r_t \in \{1, 2, \ldots, 2^{k_t} - 1\} - \{2, 2 \cdot 2, \ldots, (2^{k_t-1} - 1)2\}$, and $a_t \cdot b_t \neq 0$. There exists at least one $q \in \{\nu + \chi + \tau + \psi + 1, \ldots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi\}$ for which $a_q \cdot b_q \neq 0$. When $a_q = b_q$ then $a_q \geqslant 2$. The other indices which appear in (44), (45), (47) and (48) take the values specified in theorem 2.

(ii) The group Z_2 is contained in a non-cyclic subgroup of $C_{2^{a_t}} \times C_{2^{b_t}}$. In this case, in (44) and (45) some of the exponents a_i and $b_i, i \in \{1, \ldots, \nu + \chi + \tau + \psi\}$, or in fact all of them, can be equal to zero. Moreover, there exists precisely one $\in \{\nu + \chi + \tau + \psi + 1, \ldots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi\}$ for which $p_t = 2$ and $a_t \cdot b_t \neq 0$. When $a_t = b_t$ then $a_t \geqslant 2$. The other indices which appear in (44), (45), (47) and (48) are those assigned in theorem 2.

Alternatively, in each case, $\mathcal{C}$ can be written as in equations (42) and (54), where, in expressions (44), (45), (47) and (48) of $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2$, and $\mathcal{B}_2$ respectively, the aforementioned restrictions on the indices involved in each case have been taken into account.

This completes our description of the finite little groups of $B(2,2)$. The following conjecture (see also [18]) comments more on their significance.

Conjecture. *These finite little groups are involved in the definition of solutions to classical ultrahyperbolic general relativity which are neither self-dual nor anti-self-dual but far more general mixtures of them. These solutions correspond to the irreducibles of $B(2,2)$ which are induced from precisely these finite little groups.*

9. IRs of the little groups L_ϕ and $\mathcal{G}$-invariant measures on the orbits $\mathcal{G}/L_\phi$

From the discussion in appendix A it follows that in order to give the operators of the induced representations of $B(2, 2)$ explicitly it is necessary to give the following information:
$\mathcal{U}$. An irreducible unitary representation U of L_ϕ on a Hilbert space D for each L_ϕ.
$\mathcal{O}$. A $\mathcal{G}$-quasi-invariant measure μ on each orbit $\mathcal{G}\phi \approx \mathcal{G}/L_\phi$; where L_ϕ denotes the little group of the base point $\phi \in L_e^2(T^2)$ of the orbit $\mathcal{G}\phi \cdot \mathcal{G}$ denotes the group $G^2 = G \times G$, $G = SL(2, R)$, and $\approx$ denotes homeomorphism.

To find the induced representations of $B(2, 2) = L_e^2(T^2) Ⓢ_T G^2$, then, it is enough to provide the information cited in $\mathcal{U}$ and $\mathcal{O}$ for each of the orbit types. Following the enumeration introduced in the table of section 2 we have

$\mathcal{U}$. $\mathcal{U}_1$. The IRs U of $K = SO(2) \times SO(2)$ are parameterized by a pair of integers (n, m). For distinct representations, n and m take independently the values $n = \ldots, -2, -1, 0, 1, 2, \ldots$ and $m = \ldots, -2, -1, 0, 1, 2, \ldots$. Denoting these representations by $U^{(n,m)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(n,m)}((R(\vartheta), R(\varphi))) = \mathrm{e}^{\mathrm{i}n\vartheta}\mathrm{e}^{\mathrm{i}m\varphi}. \tag{60}$$

$\mathcal{U}_2$. The IRs U of $C_N \times SO(2)$ are parameterized by a pair of integers (ν, s) which for distinct representations take independently the values $\nu = 0, 1, 2, \ldots, N - 1$ and $s = \ldots, -2, -1, 0, 1, 2, \ldots$. Denoting these representations by $U^{(\nu,s)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(\nu,s)}(C_N \times SO(2)) = D^{(\nu,s)}\left(R\left(\frac{2\pi}{N}j\right), R(\varphi)\right) = \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}\nu j}\mathrm{e}^{\mathrm{i}s\varphi}, \tag{61}$$

where j parameterizes the elements of the group C_N.

$\mathcal{U}_3$. Similarly, the IRs U of $SO(2) \times C_N$ are parameterized by the pair of integers (t, μ), where, for distinct representations, t and μ take independently the values $t = \ldots, -2, -1, 0, 1, 2, \ldots$ and $\mu = 0, 1, 2, \ldots, N - 1$. Denoting these representations by $U^{(t,\mu)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(t,\mu)}(SO(2) \times C_N) = U^{(t,\mu)}\left(R(\vartheta), R\left(\frac{2\pi}{N}j\right)\right) = \mathrm{e}^{\mathrm{i}t\vartheta}\mathrm{e}^{\mathrm{i}\frac{2\pi}{N}\mu j}. \tag{62}$$

$\mathcal{U}_4$. We note that $H(N, p, q) = \left(R(p\vartheta), R\left(q\vartheta + \frac{2\pi}{N}j\right)\right) = H(p, q) \times C_N$, where $H(p, q) = (R(p\vartheta), R(q\vartheta))$, and, $C_N = \left(I, R\left(\frac{2\pi}{N}j\right)\right)$. We comment firstly on the IRs of $H(p, q)$. All IRs of an Abelian group are one dimensional. Let $U_{(p,q)}$ be a complex one-dimensional representation of $H(p, q)$. Then we may write

$$U_{(p,q)}((R(p\vartheta), R(q\vartheta))) = \chi_{(p,q)}(\theta)I, \tag{63}$$

where $\chi_{(p,q)} : H(p, q) \to C$ is a complex-valued function on $H(p, q)$ that is never zero, and I is the identity operator in a one-dimensional complex Hilbert space $D \approx C$. Since $U_{(p,q)}$ is a representation

$$\chi_{(p,q)}(\vartheta_1 + \vartheta_2) = \chi_{(p,q)}(\vartheta_1)\chi_{(p,q)}(\vartheta_2). \tag{64}$$

The condition for the representations being unitary reads

$$|\chi_{(p,q)}(\vartheta)| = 1. \tag{65}$$

The condition $(R(p(\vartheta + 2\pi)), R(q(\vartheta + 2\pi))) = (R(p(\vartheta)), R(q(\vartheta)))$ implies

$$\chi_{(p,q)}(\vartheta + 2\pi) = \chi_{(p,q)}(\vartheta). \tag{66}$$

Therefore, one has to find continuous complex-valued functions $\chi(\vartheta)$ satisfying the equations (64), (65) and (66). It is well known (see, for example, [26]) that all such functions have the form

$$\chi_{(p,q)}(\vartheta) = \mathrm{e}^{\mathrm{i}w\vartheta}, \qquad \text{where} \quad w \text{ is an integer.} \tag{67}$$

It is worth pointing out that $\chi_{(p,q)}(\vartheta)$ does not depend on the pair (p, q). Consequently, the IRs U of $H(p, q)$ are indexed by an integer w which for distinct representations takes the values $w = \ldots, -2, -1, 0, 1, 2, \ldots$ and are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(w)}(R(p(\vartheta)), R(q(\vartheta))) = \mathrm{e}^{\mathrm{i}w\vartheta}. \tag{68}$$

It follows that the IRs of $H(N, p, q)$ are indexed by a pair of integers (w, ξ). The indices w and ξ for distinct representations take independently the values $w = \ldots, -2, -1, 0, 1, 2, \ldots$ and $\xi = 0, 1, 2, \ldots, N - 1$. These representations, denoted by $U^{(w,\xi)}$, operate by multiplication on one complex dimension $D \approx C$ and are given by

$$U^{(w,\xi)}(H(N, p, q)) = U^{(w,\xi)}\left(\left(R\,(p\vartheta)\,, R\left(q\vartheta + \frac{2\pi}{N} j\right)\right)\right) = \mathrm{e}^{\mathrm{i}w\vartheta}\, \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}\xi j}. \tag{69}$$

$\mathcal{U}_5$. A finite little group $\mathcal{C}$ of $B(2, 2)$ is either cyclic or direct product of two cyclic groups (section 8). The cyclic little groups are described in detail in theorem 4 and the non-cyclic ones are given explicitly in theorem 5. When $\mathcal{C}$ is cyclic (equation (57)) the IRs U of $\mathcal{C}$ are indexed by an integer κ which for distinct representations takes values in the set $\{0, \ldots, |\mathcal{C}| - 1\}$. The order $|\mathcal{C}|$ of $\mathcal{C}$ is given by (58). These representations, denoted by $U^{(\kappa)}$, are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(\kappa)}(\mathcal{C}) = U^{(\kappa)}\left(\left(R\left(\left(\frac{2\pi}{n}\mathcal{A}\right)i\right), R\left(\left(\frac{2\pi}{m}\mathcal{B}\right)i\right)\right)\right) = \mathrm{e}^{\mathrm{i}\frac{2\pi}{|\mathcal{C}|}\kappa i}. \tag{70}$$

When $\mathcal{C}$ is a direct product of two cyclic groups, i.e., $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, (equation (54)) the IRs of $\mathcal{C}$ are indexed by a pair of integers (γ, δ). The indices γ and δ for distinct representations take independently the values $\gamma = 0, 1, 2, \ldots, |\mathcal{C}_1| - 1$ and $\delta = 0, 1, 2, \ldots, |\mathcal{C}_2| - 1$, where, $|\mathcal{C}_1|$ and $|\mathcal{C}_2|$ are given by (53). These representations, denoted by $U^{(\gamma,\delta)}$, operate on one complex dimension $D \approx C$ and are given by (equation (54))

$$\begin{aligned} U^{(\gamma,\delta)}(\mathcal{C}) &= U^{(\gamma,\delta)}\left(\left(R\left(\frac{2\pi}{n}(\mathcal{A}_1 j_1 + \mathcal{A}_2 j_2)\right), R\left(\frac{2\pi}{m}(\mathcal{B}_1 j_1 + \mathcal{B}_2 j_2)\right)\right)\right) \\ &= \mathrm{e}^{\mathrm{i}\frac{2\pi}{|\mathcal{C}_1|}\gamma j_1}\, \mathrm{e}^{\mathrm{i}\frac{2\pi}{|\mathcal{C}_2|}\delta j_2}, \end{aligned} \tag{71}$$

where $j_1 \in \{0, \ldots, |\mathcal{C}_1| - 1\}$ and $j_2 \in \{0, \ldots, |\mathcal{C}_2| - 1\}$. We now proceed to give the information cited in $\mathcal{O}$. Although a $\mathcal{G}$-quasi-invariant measure is all what is needed, a $\mathcal{G}$-invariant measure will be provided in all cases. Following the enumeration introduced in the table of section 2 we have

$\mathcal{O}$. $\mathcal{O}_1$. In appendix B it is proved that the construction of a unique (up to a constant factor) $\mathcal{G}$-invariant measure on the orbits $01 \equiv \mathcal{G}/L_\phi$, $L_\phi = K = SO(2) \times SO(2)$ necessitates the construction of a $\mathcal{G}$-invariant measure on $\mathcal{G}$ and the construction of a K-invariant measure on K. A $\mathcal{G}$-invariant measure on

$$\mathcal{G} = \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ j & k \end{pmatrix}\right), a, b, c, d, e, f, j, k \in R, ad - bc = 1, ek - jf = 1\right\}$$

is given [27] by the 6-form

$$\mathrm{d}g = \frac{\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c \wedge \mathrm{d}e \wedge \mathrm{d}f \wedge \mathrm{d}j}{ae} \tag{72}$$

Thus, an invariant measure on $\mathcal{G}$ is obtained. A K-invariant measure on K is given by the 2-form $\mathrm{d}\theta \wedge \mathrm{d}\phi$, where θ, ϕ are the usual angular coordinates which cover the 2-torus $SO(2) \times SO(2) \approx S^1 \times S^1$.

$\mathcal{O}_2$. Regarding the orbits $02 \equiv \mathcal{G}/L_\phi$, $L_\phi = C_N \times SO(2)$, we note that $02 \equiv \mathcal{G}/(C_N \times SO(2)) = (\mathcal{G}/(I \times SO(2)))/(C_N \times I)$. In appendix B a $\mathcal{G}$-invariant measure on the quotient spaces $\widetilde{02} \equiv \mathcal{G}/(I \times SO(2))$ is given. The orbits 02 inherit the constructed $\mathcal{G}$-invariant measure on the spaces $\widetilde{02}$. The reason is the following. The quotient space $02 \equiv \widetilde{02}/(C_N \times I)$ is precisely the space of orbits of the right action $T_2 : \widetilde{02} \times (C_N \times I) \longrightarrow \widetilde{02}$ of the group $C_N \times I$ on $\widetilde{02}$ defined by

$$((g,h)(I \times SO(2))) \circ \left(R\left(\frac{2\pi}{N}i\right) \times I\right) := \left((g,h) \cdot \left(R\left(\frac{2\pi}{N}i\right) \times I\right)\right)(I \times SO(2)), \tag{73}$$

where $(g,h) \in \mathcal{G}$, $(g,h)(I \times SO(2)) \in \widetilde{02}$, and $R\left(\frac{2\pi}{N}i\right) \in C_N$. The symbol $\circ$ denotes the action T_2, and $\cdot$ denotes the group multiplication in $\mathcal{G}$. It can be easily proved that the action (73) is fixed point free. Since C_N is finite and since the action (73) is fixed point free the orbits 02 inherit the $\mathcal{G}$-invariant measure on the quotient spaces $\widetilde{02}$.

$\mathcal{O}_3$. Regarding the orbits $03 \equiv \mathcal{G}/L_\phi$, $L_\phi = SO(2) \times C_N$, we note that $03 \equiv \mathcal{G}/(SO(2) \times C_N) = (\mathcal{G}/(SO(2) \times I))/(I \times C_N)$. The construction of a $\mathcal{G}$-invariant measure on the orbits 03 is similar to the construction in case $\mathcal{O}_2$.

$\mathcal{O}_4$. Regarding the orbits $04 \equiv \mathcal{G}/L_\phi$, $L_\phi = H(N,p,q) = H(p,q) \times (I \times C_N)$, where $H(p,q) = (R(p\vartheta), R(q\vartheta))$, we note that $04 \equiv \mathcal{G}/H(N,p,q) = (\mathcal{G}/H(p,q))/(I \times C_N)$. In appendix B a $\mathcal{G}$-invariant measure on the quotient spaces $\widetilde{04} \equiv \mathcal{G}/H(p,q)$ is given. The orbits 04 inherit the aforementioned $\mathcal{G}$-invariant measure on the quotient spaces $\widetilde{04}$: the orbit $04 \equiv \widetilde{04}/(I \times C_N)$ is precisely the space of orbits of the right action $T_4 : \widetilde{04} \times (I \times C_N) \longrightarrow \widetilde{04}$ of the group $I \times C_N$ on $\widetilde{04}$ defined by

$$((g,h)H(p,q)) * \left(I \times R\left(\frac{2\pi}{N}i\right)\right) := \left((g,h) \cdot \left(I \times R\left(\frac{2\pi}{N}i\right)\right)\right) H(p,q), \tag{74}$$

where $(g,h) \in \mathcal{G}$, $(g,h)H(p,q) \in \widetilde{04}$, and $R\left(\frac{2\pi}{N}i\right) \in C_N$. The symbol $*$ denotes the action T_4. Since C_N is finite and since the action (74) is fixed point free the orbits 04 inherit the $\mathcal{G}$-invariant measure on the quotient spaces $\widetilde{04}$.

$\mathcal{O}_5$. The orbits $05 \equiv \mathcal{G}/L_\phi$, $L_\phi = \mathcal{C}$, where $\mathcal{C}$ is either cyclic or direct product of two cyclic groups, can be endowed with the $\mathcal{G}$-invariant measure on $\mathcal{G}$ given in case $\mathcal{O}_1$. Indeed, the orbit $05 \equiv \mathcal{G}/L_\phi$ is the space of orbits of the right action $T_5 : \mathcal{G} \times \mathcal{C} \longrightarrow \mathcal{G}$ of the group $\mathcal{C}$ on $\mathcal{G}$ given by

$$(g,h) \star c := (g,h) \cdot c, \tag{75}$$

where $(g,h) \in \mathcal{G}$, and $c \in \mathcal{C}$. Thus the action T_5 denoted by $\star$ is identical to the group multiplication in $\mathcal{G}$. Since the group $\mathcal{C}$ is finite and since the action (75) is fixed point free the coset space $05 \equiv \mathcal{G}/\mathcal{C}$ inherits the measure on $\mathcal{G}$.

This completes the necessary information in order to construct the induced representations of $B(2,2)$. The following remarks are in order regarding the representations of $B(2,2)$ obtained by the above construction.

The subgroup $L_e^2(T^2)$ of $B(2,2) = L_e^2(T^2) Ⓢ_T G^2$ has been topologized as a (pre)Hilbert space by using a natural measure $\mathrm{d}\theta \wedge \mathrm{d}\phi$ on the 2-torus $T^2 = S^1 \times S^1$ and by introducing a scalar product into $L_e^2(T^2)$ defined by (9). The Hilbert-type topology employed here for $L_e^2(T^2) \times G^2$ is the appropriate one [28] for describing quantum mechanical systems in asymptotically flat spacetimes. The group $G = SL(2,R)$ has been endowed with the standard topology, and $B(2,2)$ in the product topology of $L_e^2(T^2)\times G^2$ becomes a non-locally compact group; the proof follows without substantial change in Cantoni's proof [6] for the ordinary BMS group B. Since in the Hilbert-type topology $B(2,2)$ is not locally compact the theorems dealing with the irreducibility of the representations obtained by the above construction no longer apply [8]. However, the induced representations obtained above *are* irreducible. The proof follows very closely the one given in [13] for the case of the original BMS group B. If the $\mathcal{G}$-action on the dual group $\widehat{L}$ of $L_e^2(T^2)$ is not too pathological then the list of irreducibles obtained by the above construction is exhaustive. The $\mathcal{G}$-action is not too pathological when the $\mathcal{G}$-orbits in $\widehat{L}$ can be enumerated in some way, namely, when there is a Borel set in $\widehat{L}$ that meets each orbit exactly once. Interestingly enough, it has been proved [32, 33] that this is precisely the case in the Hilbert-type topology employed here. To conclude, in this paper *all* the continuous, unitary, irreducible representations of $B(2,2)$ have been constructed.

10. Conclusion

The main results and conclusions obtained in the present paper are listed here:

(i) The finite little groups of $B(2,2)$ have been determined explicitly and the operators of the $B(2,2)$-irreducibles have been given in all cases by using Wigner–Mackey's theory of induced representations of semi-direct products. The list of irreducibles so obtained is exhaustive because the 'supertranslations' $L_e^2(T^2)$ of $B(2,2)$ have been employed with the Hilbert topology. Quite unexpectedly, the representation theory dictates that *all* the $B(2,2)$-elementary entities—apart from those induced from cyclic little groups—carry two distinct discrete 'spins' which take integer values in all cases.

(ii) The results presented here will also be useful in the study of the representation theory of *CB* and *EB* since subgroups of $C_n\times C_m$ are expected [17] to appear as little groups of *CB* and *EB*. These two groups figure prominently among the 42 generalizations defined in [17]. The study of the representation theory of *CB* is important in this research programme since *CB* is the complexification of B as well as of $B(2,2)$, and the study of the representation theory of *EB* will clarify [18] the connection with the ALE gravitational instantons.

(iii) In the case of *EB* the large number of subgroups of $C_n\times C_m$ suggests [18] that the gravitational multi-instantons of Gibbons and Hawking [31] represent only a very small number of solutions of a class of solutions whose more general members are mixtures of self-dual and anti-self-dual solutions. The existence of more general solutions in the Euclidean case has also been suggested in a different context by G't Hooft [25].

(iv) It has been proved [14, 15] that *all* ALE spaces at infinity resemble a quotient R^4/Γ, where Γ is a finite subgroup of $SU(2)$. The research programme being pursued here suggests [18] that more general solutions which are mixtures of self-dual and anti-self-dual solutions exist both in real spacetimes in Euclidean and ultrahyperbolic signatures and in complex spacetimes. In the ultrahyperbolic case the finite little groups determined here are expected to play the role of Γ in the Euclidean case.

(v) In these more general solutions, subgroups of $C_n\times C_m$—the finite little groups in each case—are expected [18] to play the role of Γ both in the Euclidean and in the complex case. The physical interpretation of these odd-looking solutions—flat manifolds whose certain points are identified at the neighbourhood of infinity—which do not seem to have analogues in other areas of physics and their significance for low-energy quantum gravity is an open problem in all cases.

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Appendix A. Summary of Wigner–Mackey's representation theory of semi-direct products

We will give the bare essentials of Wigner–Mackey's representation theory ([7–10, 29]) in order to construct explicitly the operators of the induced representations of $B(2, 2)$.

Let A and $\mathcal{G}$ be topological groups, and let T be a given homomorphism from $\mathcal{G}$ into the group of automorphisms $\mathrm{Aut}(A)$ of A. Suppose that A is Abelian and $\mathcal{B} = A \circledS_T \mathcal{G}$ is the semi-direct product of A and $\mathcal{G}$ is specified by the continuous action $T : \mathcal{G} \longrightarrow \mathrm{Aut}(A)$. In the product topology of $A \times \mathcal{G}, \mathcal{B}$ then becomes a topological group. It is assumed that it becomes a separable locally compact topological group.

The irreducible continuous representations of A (characters) can be given in the structure of an Abelian group $\hat{A}$, the dual group of A, with group operation given by $(\chi_1\chi_2)(\alpha) = \chi_1(\alpha)\chi_2(\alpha)$. Any bijective map $\mu : A \longrightarrow A$ induces a map $\hat{A} \longrightarrow \hat{A}$, $\chi \longrightarrow \mu\chi$, defined by

$$(\mu\chi)(\alpha) := \chi(\mu^{-1}\alpha). \tag{A.1}$$

In this way the action T of $\mathcal{G}$ on A induces a dual action $\widehat{T}$ of $\mathcal{G}$ on $\hat{A}$ defined by

$$(\widehat{T}(g)\chi)(\alpha) := \chi(T(g^{-1})\alpha), \tag{A.2}$$

where $g \in \mathcal{G}$, $\chi \in \hat{A}$, and $\alpha \in A$. Since $T(g)\alpha = g\alpha g^{-1}$ the last equation yields

$$(\widehat{T}(g)\chi)(\alpha) = \chi(g^{-1}\alpha g). \tag{A.3}$$

For a given character $\chi \in \hat{A}$, the *largest* subgroup L_χ of $\mathcal{G}$ which leaves χ fixed is called the little group of χ, i.e.,

$$L_\chi = \{g \in \mathcal{G} \mid \widehat{T}(g)\chi = \chi\}. \tag{A.4}$$

L_χ is a closed subgroup of $\mathcal{G}$. The set of characters which can be reached from χ by the $\mathcal{G}$-action is called the orbit of χ, denoted by $\mathcal{G}\chi$. Since A acts trivially on A and hence on $\hat{A}$ (equation (A.3)), the largest subgroup of $\mathcal{B}$ which leaves χ fixed is the semi-direct product $\mathcal{B}_\chi = A \circledS_T L_\chi$. $\mathcal{B}_\chi$ is a closed subgroup of $\mathcal{B}$.

In the class of groups we are considering, we restrict our attention to measures on $\hat{A}$ which are concentrated on single orbits of the $\mathcal{G}$-action $\widehat{T}$. The remaining ergodic measures not concentrated on single orbits are called strictly ergodic. Now there is a natural bijection

$$\mathcal{G}/L_\chi \longrightarrow \mathcal{G}\chi,$$

given by $gL_\chi \longrightarrow g\chi$, where $g \in \mathcal{G}$. This bijection will even be a homeomorphism under quite general conditions. We will identify that $\mathcal{G}/L_\chi$ and $\mathcal{G}\chi \cdot \hat{A}$ is a disjoint union of the orbits $\mathcal{G}\chi$. The trivial action of A on $\hat{A}$ implies that $\mathcal{G}\chi = \mathcal{B}\chi$ and hence $\mathcal{G}/L_\chi = \mathcal{B}/\mathcal{B}_\chi$. Let U be a continuous irreducible unitary representation of the little group L_χ on a Hilbert space D. Then

$$\chi U : (\alpha, l) \longmapsto \chi(a)U(l), \tag{A.5}$$

where $\alpha \in A$ and $l \in L_\chi$ is a continuous unitary irreducible representation of the group $\mathcal{B}_\chi = A \circledS_T L_\chi$ in D.

Let $\mathcal{X}$ be a topological space. Two measures $\nu, \widetilde{\nu}$ on $\mathcal{X}$ are equivalent if they assume the value zero for the same Borel sets in $\mathcal{X}$. The measure class of ν is its equivalence class under this equivalence relation. Since L_χ is a closed subgroup of $\mathcal{G}$ and $\mathcal{G}$ is locally compact there is [29] a unique non-zero measure class $\mathcal{M}$ in the coset space $\mathcal{G}/L_\chi$, called the *invariant measure class* on $\mathcal{G}/L_\chi$, such that for each $\nu \in \mathcal{M}$ and each $g \in \mathcal{G}$ the measure ν_g defined by

$$\nu_g(E) := \nu(g^{-1}E), \quad \text{for each Borel set } E \subset \mathcal{G}/L_\chi, \tag{A.6}$$

is also in $\mathcal{M}$. If for all $g \in \mathcal{G}$, the measures ν_g and ν have the same Borel sets of measure zero then the measure ν is called $\mathcal{G}$-quasi-invariant. The measure ν is called $\mathcal{G}$-invariant when

$$\nu_g = \nu, \quad \text{for each } g \in \mathcal{G}. \tag{A.7}$$

When ν is $\mathcal{G}$-quasi-invariant there exists a positive (ν-a.e.) continuous function on $\mathcal{G}/L_\chi$, denoted by $\mathrm{d}\nu_g/\mathrm{d}\nu$, such that

$$\nu_g[E] = \int_E \left(\frac{\mathrm{d}\nu_g}{\mathrm{d}\nu}\right)(p)\,\mathrm{d}\nu(p) \quad \text{for all Borel sets } E \subset \mathcal{G}/L_\chi. \tag{A.8}$$

The 'Jacobian' $\mathrm{d}\nu_g/\mathrm{d}\nu$ is known as the Radon–Nikodym derivative of ν_g with respect to ν. Let $\mathcal{H}_\nu$ denote the space of functions $\psi : \mathcal{G} \longrightarrow D$ which satisfy the conditions

$$\text{(a)} \quad \psi(gh) = U(h^{-1})\psi(g) \quad (g \in \mathcal{G}, h \in L_\chi), \tag{A.9}$$

$$\text{(b)} \quad \int_{\mathcal{G}\chi} \langle \psi(p), \psi(p)\rangle\,\mathrm{d}\nu(p) < \infty, \tag{A.10}$$

where the scalar product under the integral sign is that of D, and, in (b), the integrand is expressed as a function on $\mathcal{G}\chi \approx \mathcal{G}/L_\chi$ since, in view of (a), the integrand is constant on the cosets in $\mathcal{G}/L_\chi$. $\mathcal{H}_\nu$ is turned into a Hilbert space by introducing the scalar product

$$\langle \psi_1, \psi_2\rangle = \int_{\mathcal{G}\chi} \langle \psi_1(p), \psi_2(p)\rangle\,\mathrm{d}\nu(p). \tag{A.11}$$

Define an action of $\mathcal{B} = A \circledS_T \mathcal{G}$ on $\mathcal{H}_\nu$ by

$$(g_\mathrm{o}\psi)(g) = \sqrt{\frac{\mathrm{d}\nu_{g_\mathrm{o}}}{\mathrm{d}\nu}(g\chi)}\,\psi(g_\mathrm{o}^{-1}g), \tag{A.12}$$

$$(\alpha\psi)(g) = [(\widehat{T}(g)\chi)(\alpha)]\psi(g), \tag{A.13}$$

where $g, g_\mathrm{o} \in \mathcal{G}$, $\chi \in \hat{A}$, $\alpha \in A$, and $\mathrm{d}\nu_{g_\mathrm{o}}\big/\mathrm{d}\nu$ is the Radon–Nikodym derivative of ν_{g_o} with respect to ν. It is straightforward to show that this action gives a unitary representation of B on $\mathcal{H}_\nu$, which is continuous whenever U is. This is the representation of B *induced* from χ and the irreducible representation U of the little group L_χ. We note that when ν is $\mathcal{G}$-invariant then $\frac{\mathrm{d}\nu_{g_\mathrm{o}}}{\mathrm{d}\nu}(g) = 1$, and this is precisely what happens in the case of $B(2, 2)$.

In a nutshell the central results of induced representation theory are the following:

(i) Given the topological restrictions on $\mathcal{B} = A \circledS_T \mathcal{G}$ (separability and local compactness), any representation of $\mathcal{B}$, constructed by the method above, is irreducible if the representation U of L_χ on D is irreducible. Thus an irreducible representation of $\mathcal{B}$ is obtained for *each* $\chi \in \hat{A}$ and *each* irreducible representation U of L_χ.

(ii) If $\mathcal{B} = A \circledS_T \mathcal{G}$ is a regular semi-direct product (i.e., $\hat{A}$ contains a Borel subset which meets each orbit in $\hat{A}$ under $\mathcal{B}$ in just one point) then *all* of its irreducible representations can be obtained in this way.

Appendix B. $\mathcal{G}$-invariant measures on the quotient spaces 01, $\widetilde{02}$, $\widetilde{03}$, $\widetilde{04}$

To provide $\mathcal{G}$-invariant measures on the homogeneous spaces 01, $\widetilde{02}$, $\widetilde{03}$, $\widetilde{04}$ it is convenient to employ the following ([30]):

Theorem. *Let $\mathcal{G}$ be a Lie group, H a closed subgroup. The relation*

$$|\det \mathrm{Ad}_{\mathcal{G}}(h)| = |\det \mathrm{Ad}_{H}(h)| \quad (h \in H) \tag{B.1}$$

is a necessary and sufficient condition for the existence of a positive $\mathcal{G}$-invariant measure $\mathrm{d}\mu_H$ on $\mathcal{G}/H$, which is unique up to a constant factor, and satisfies

$$\int_{\mathcal{G}} f(g)\,\mathrm{d}\mu(g) = \int_{\mathcal{G}/H} \left(\int_H f(gh)\,\mathrm{d}\mu(h) \right) \mathrm{d}\mu_H, \tag{B.2}$$

where $\mathrm{d}\mu(g)$ and $\mathrm{d}\mu(h)$ are, respectively, suitably normalized invariant measures on $\mathcal{G}$ and H.

Here $\mathrm{Ad}_{\mathcal{G}}$ denotes the adjoint representation of the group $\mathcal{G}$ and f is any continuous function of compact support on $\mathcal{G}$. Since any function on $\mathcal{G}$ constant on H cosets may be regarded as a function on $\mathcal{G}/H$, (B.2) *defines* $\mathrm{d}\mu_H$ on $\mathcal{G}/H$. Thus, in each case, it is sufficient to verify the condition on the moduli of the determinants, and to provide an H-invariant measure $\mathrm{d}\mu(h)$ on H. H-invariant measures are given in section 9. The condition (B.1) on the determinants is verified for the case of the homogeneous space $\widetilde{04} \equiv \mathcal{G}/H(p, q)$. The other cases are similar.

Since $H = H(p, q)$ is Abelian, $\mathrm{Ad}_{H(p,q)}(\varepsilon)$ is the identity operator, so that $|\det \mathrm{Ad}_{H(p,q)}(\varepsilon)| = 1$ for all $\varepsilon \in H(p, q)$. It must now be shown that $|\det \mathrm{Ad}_{\mathcal{G}}(\varepsilon)| = 1$. A basis for the Lie algebra of $\mathcal{G}$ is given by the generators

$$\Omega_i = \begin{bmatrix} A_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \Omega_{3+i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_i \end{bmatrix}, \tag{B.3}$$

where $\mathbf{0}$ is the 2×2 zero-matrix and A_i, $i = 1, 2, 3$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{B.4}$$

is a basis for the Lie algebra of $SL(2, R)$. Taking

$$\varepsilon = \begin{bmatrix} \cos(p\theta) & \sin(p\theta) & 0 & 0 \\ -\sin(p\theta) & \cos(p\theta) & 0 & 0 \\ 0 & 0 & \cos(q\theta) & \sin(q\theta) \\ 0 & 0 & -\sin(q\theta) & \cos(q\theta) \end{bmatrix}, \tag{B.5}$$

a straightforward calculation gives

$$\mathrm{Ad}_{\mathcal{G}}(\varepsilon) = \begin{bmatrix} \cos(2p\theta) & \sin(2p\theta) & 0 & 0 & 0 & 0 \\ -\sin(2p\theta) & \cos(2p\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(2q\theta) & \sin(2q\theta) & 0 \\ 0 & 0 & 0 & -\sin(2q\theta) & \cos(2q\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B.6}$$

Evidently, $\det \mathrm{Ad}_{\mathcal{G}}(\varepsilon) = 1$ for all $\varepsilon \in H(p, q)$, so that the required condition (B.1) is satisfied. Hence the unique (up to a constant factor) $\mathcal{G}$-invariant measure on $\widetilde{04} \equiv \mathcal{G}/H(p, q)$ is given.

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